

SHOCK DIFFRACTION BY CONVEX CORNERED WEDGES FOR THE NONLINEAR WAVE SYSTEM

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ABSTRACT. We are concerned with rigorous mathematical analysis of shock diffraction by two-dimensional convex cornered wedges in compressible fluid flow governed by the nonlinear wave system. This shock diffraction problem can be formulated as a boundary value problem for second-order nonlinear partial differential equations of mixed elliptic-hyperbolic type in an unbounded domain. It can be further reformulated as a free boundary problem for nonlinear degenerate elliptic equations of second order. We establish a first global theory of existence and regularity for this shock diffraction problem. In particular, we establish that the optimal regularity for the solution is $C^{0,1}$ across the degenerate sonic boundary. To achieve this, we develop several mathematical ideas and techniques, which are also useful for other related problems involving similar analytical difficulties.

1. INTRODUCTION

We are concerned with rigorous mathematical analysis of shock diffraction by two-dimensional cornered wedges whose angles are less than π in compressible fluid flow governed by the nonlinear wave system. The study of shock diffraction problems can date back 1950's by the work of Bargman [3], Lighthill [24, 25], Fletcher-Weimer-Bleakney [14], and Fletcher-Taub-Bleakney [13] via asymptotic or experimental analysis. Also Courant-Friedrichs [10] and Whitham [27].

In this paper, we develop several mathematical ideas and techniques through the nonlinear wave system to establish a rigorous theory of existence and regularity of solutions to the diffraction problem. The nonlinear wave system consists of three conservation laws, which takes the form:

$$(1.1) \quad \begin{aligned} \rho_t + m_{x_1} + n_{x_2} &= 0, \\ m_t + p_{x_1} &= 0, \\ n_t + p_{x_2} &= 0, \end{aligned}$$

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for $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$, $\mathbf{x} \in \mathbb{R}^2$, where ρ stands for the density, p for the pressure, (m, n) for the momenta in the (x_1, x_2) -coordinates. The pressure-density constitutive relation is

$$(1.2) \quad p(\rho) = \rho^\gamma / \gamma, \quad \gamma > 1,$$

by scaling without loss of generality. Then the sonic speed $c = c(\rho)$ is determined by

$$c^2(\rho) := p'(\rho) = \rho^{\gamma-1}.$$

Notice that $c(\rho)$ is a positive, increasing function for all $\rho > 0$.

The two-dimensional nonlinear wave system (1.1) is derived from the compressible isentropic gas dynamics by neglecting the inertial terms, i.e., the quadratic terms in the velocity; see Canic-Keyfitz-Kim [5]. Also see Zheng [28] for a related hyperbolic system, the pressure gradient system of conservation laws; the same arguments developed in this paper can be carried through to establish a corresponding theory of existence and regularity for the pressure gradient system.

Let S_0 be the vertical planar shock in the (t, \mathbf{x}) -coordinates, $t \in \mathbb{R}_+ := [0, \infty)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with the left constant state $U_1 = (\rho_1, m_1, 0)$ and the right state $U_0 = (\rho_0, 0, 0)$, satisfying

$$m_1 = \sqrt{(p(\rho_1) - p(\rho_0))(\rho_1 - \rho_0)} > 0, \quad \rho_1 > \rho_0.$$

When S_0 passes through a convex cornered wedge:

$$W := \{(x_1, x_2) : x_2 < 0, -\infty < x_1 \leq x_2 \tan \theta_w\},$$

shock diffraction occurs, where the wedge angle θ_w is between $-\pi$ and 0; see Fig. 1. Then the shock diffraction problem can be formulated as the following mathematical problem:

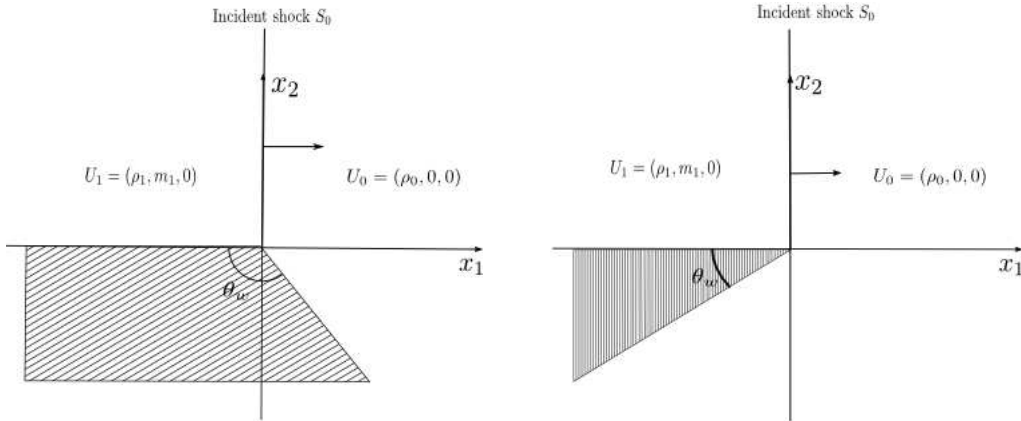


FIGURE 1. Initial-boundary value problem

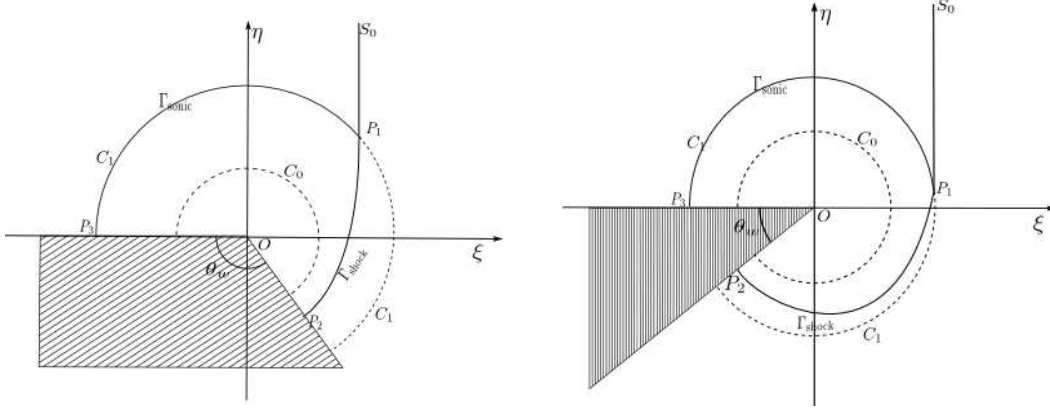


FIGURE 2. Shock diffraction configuration

Problem 1 (Initial-Boundary Value Problem). *Seek a solution of system (1.1) with the initial condition at $t = 0$:*

$$(1.3) \quad (\rho, m, n)|_{t=0} = \begin{cases} (\rho_0, 0, 0) & \text{in } \{x_1 > 0, x_2 > 0\} \cup \{(x_2 - x_1 \tan \theta_w)x_1 \geq 0, x_2 < 0\}, \\ (\rho_1, m_1, 0) & \text{in } \{x_1 < 0, x_2 > 0\}, \end{cases}$$

and the slip boundary condition along the wedge boundary ∂W :

$$(1.4) \quad (m, n) \cdot \boldsymbol{\nu}|_{\partial W} = 0,$$

where $\boldsymbol{\nu}$ is the exterior unit normal to ∂W (see Fig. 1).

Notice that the initial-boundary value problem (1.1)–(1.4) is invariant under the self-similar scaling:

$$(1.5) \quad (t, \mathbf{x}) \rightarrow (\alpha t, \alpha \mathbf{x}) \quad \text{for } \alpha \neq 0.$$

Thus, we seek self-similar solutions with the form

$$(1.6) \quad (\rho, m, n)(t, \mathbf{x}) = (\rho, m, n)(\xi, \eta) \quad \text{for } (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

In the self-similar coordinates (ξ, η) , system (1.1) can be rewritten as

$$(1.7) \quad \begin{aligned} (m - \xi\rho)_\xi + (n - \eta\rho)_\eta + 2\rho &= 0, \\ (p(\rho) - \xi m)_\xi - (\eta m)_\eta + 2m &= 0, \\ (\xi n)_\xi - (p(\rho) - \eta n)_\eta - 2n &= 0. \end{aligned}$$

In the polar coordinates (r, θ) , $r = \sqrt{\xi^2 + \eta^2}$, the system can be further written as

$$(1.8) \quad \partial_r \begin{pmatrix} r\rho - \cos \theta m - \sin \theta n \\ rm - \cos \theta p(\rho) \\ rn - \sin \theta p(\rho) \end{pmatrix} + \partial_\theta \begin{pmatrix} \sin \theta m - \cos \theta n \\ \sin \theta p(\rho) \\ -\cos \theta p(\rho) \end{pmatrix} = \begin{pmatrix} \rho + \frac{\cos \theta}{r} m + \frac{\sin \theta}{r} n \\ m + \frac{\cos \theta}{r} p(\rho) \\ n + \frac{\sin \theta}{r} p(\rho) \end{pmatrix}.$$

The location of the incident shock S_0 for large $r \gg 1$ is:

$$(1.9) \quad \xi = \xi_1 = \sqrt{\frac{p(\rho_1) - p(\rho_0)}{\rho_1 - \rho_0}} > 0.$$

Then Problem 1 can be reformulated as a boundary value problem in an unbounded domain:

Problem 2 (Boundary Value Problem). *Seek a solution of system (1.7), or equivalently (1.8), with the asymptotic boundary condition when $r \rightarrow \infty$:*

$$(1.10) \quad (\rho, m, n) \rightarrow \begin{cases} (\rho_0, 0, 0) & \text{in } \{\xi > \xi_1, \eta > 0\} \cup \{(\eta - \xi \tan \theta_w)\xi \geq 0, \eta < 0\}, \\ (\rho_1, m_1, 0) & \text{in } \{\xi < \xi_1, \eta > 0\}, \end{cases}$$

and the slip boundary condition along the wedge boundary ∂W :

$$(1.11) \quad (m, n) \cdot \boldsymbol{\nu}|_{\partial W} = 0,$$

where $\boldsymbol{\nu}$ is the exterior unit normal to ∂W (see Fig. 2).

For a smooth solution $U = (\rho, m, n)$ to (1.7), we may eliminate m and n in (1.1) to obtain a second-order nonlinear equation for ρ :

$$(1.12) \quad ((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta + \xi\rho)_\xi + ((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi + \eta\rho)_\eta - 2\rho = 0.$$

Correspondingly, equation (1.12) in the polar coordinates (r, θ) , $r = \sqrt{\xi^2 + \eta^2}$, takes the form

$$(1.13) \quad ((c^2 - r^2)\rho_r)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0.$$

In the self-similar coordinates, as the incident shock S_0 passes through the wedge corner, S_0 interacts with the sonic circle Γ_{sonic} of state (1): $r = r_1$, and becomes a transonic diffracted shock Γ_{shock} , and the flow in the domain Ω behind the shock and inside Γ_{sonic} becomes subsonic. In Section 2, we reduce *Problem 2* for shock diffraction into a one-phase free boundary problem, *Problem 3*, for second-order elliptic equation in the domain Ω with the free boundary Γ_{shock} , degenerate boundary Γ_{sonic} , and slip boundary $\partial W \cap \Omega$. In this paper, we focus on the existence of global solutions of shock diffraction and the optimal regularity of the solution across the sonic circle Γ_{sonic} .

There are three main difficulties to establish the global existence of solutions. The first is that the ellipticity degenerates at the sonic circle Γ_{sonic} . The second is that the oblique derivative boundary condition degenerates at P_2 . The third difficulty is that the diffracted shock may coincide with the sonic circle $C_0 := \{r = c(\rho_0)\}$ of state (0) in the iteration where the oblique derivative condition fails, that is, β_2 may equal to 0. Then we can not employ directly the results in Liebermann [20]–[23] to show the existence of solutions for the fixed boundary value problem. One of our strategies here is to add an additional condition $r(\theta) \geq c(\rho_0) + \delta$ on the diffracted

shock curve with δ small enough and modify slightly the approximate shock curve to overcome the difficulty.

The approach used in this paper for establishing the global existence of solutions is first to regularize the equation by adding the regularized differential operator $\varepsilon\Delta$ to make the equation uniformly elliptic; and then to rely on the Perron method, as in [19], to show the global existence of solutions for the fixed boundary value problem; and finally to apply the Schauder fixed point theorem to show the existence of global solutions for the free boundary problem. Moreover, we obtain uniform estimates for the global solutions with respect to $\delta, \varepsilon > 0$ so that we can pass the limits $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ to establish the existence of solutions of the free boundary problem for the original system. In particular, we prove that the diffracted shock is uniformly transonic, that is, the strength of the shock is positive even at the point P_2 .

In order to establish the optimal regularity across the sonic boundary Γ_{sonic} , we write equation (1.13) in terms of the function

$$\psi := c^2(\rho_1) - c^2(\rho)$$

in the (x, y) -coordinates, which is specified in §5, defined near Γ_{sonic} such that Γ_{sonic} becomes a segment on $\{x = 0\}$, with the form

$$(2c_1x - \psi)\psi_{xx} + \psi_{yy} + c_1\psi_x - \psi_x^2 - \frac{1}{(\gamma - 1)c_1^2}\psi_y^2 = 0 \quad \text{in } x > 0 \text{ and near } x = 0, \quad (1.14)$$

plus “small” terms, since ρ and ψ have the same regularity in Ω . Also $\psi > 0$ in $\{x > 0\}$ and $\psi = 0$ on $\{x = 0\}$. For our solution ψ , (1.14) is elliptic in $\{x > 0\}$. It is easy to check that the solution ρ is Lipschitz continuous up to the sonic circle $\Gamma_{\text{sonic}} = \{r = r_1\} \cap \partial\Omega$.

Let $W = \psi - c_1x$. Then ψ and W have the same regularity near $x = 0$. We consider the Dirichlet boundary value problem for W . We employ the approach in Bae-Chen-Feldman [2] to analyze the features of equation (1.14) and prove the $C^{1,\alpha}$ -regularity of solutions of the shock diffraction problem in the elliptic region up to the part $\overline{\Gamma_{\text{sonic}}} \setminus P_1$ of the sonic shock. As a corollary, we establish that the $C^{0,1}$ -regularity is actually optimal across the sonic boundary Γ_{sonic} from the elliptic region Ω to the hyperbolic region of state (1), that is, the optimal regularity at the degenerate elliptic boundary.

We remark that the existence problem for a shock interaction with the right cornered wedge (90-degree) was studied by Kim [18], in which some important features and behavior of solutions have been exhibited. As far as we have known, for the shock diffraction by a convex cornered wedges whose angles are between $-\pi$ and 0 in compressible fluid flow, no rigorous complete global mathematical results have been available, since the early work by Bargman [3], Lighthill [24, 25], Fletcher-Taub-Bleakney [13], and Fletcher-Weimer-Bleakney [14]. The results established in this paper is the first rigorous complete mathematical results through the nonlinear wave system for the global existence and optimal regularity of solutions of shock diffraction by any convex cornered wedge.

A closely related problem, shock reflection-diffraction by a concave cornered wedges, has been systematically analyzed in Chen-Feldman [6, 7, 8], where the existence of regular shock reflection-diffraction configurations has been established up to the sonic wedge-angle for potential flow. Also see Canic-Keyfitz-Kim [4, 5] for the unsteady transonic small disturbance equation and the nonlinear wave system, and Zheng [28] for the pressure-gradient system.

The organization of this paper is as follows. In §2, we reformulate the shock diffraction problem into a free boundary problem for the nonlinear second-order equation (1.1) in both the self-similar and polar coordinates, and present the statement of our main theorem for the existence and optimal regularity of the global solution. In §3, we first formulate the regularized approximate free boundary problem by adding a regularized differential operator with $\varepsilon\Delta\rho$ to the original equation (Δ denotes the Laplace operator in the self-similar coordinates) and the assumption $c(\bar{\rho}) \geq c(\rho_0) + \delta$, where $\bar{\rho}$ is the data given at the point P_2 . Then we establish the existence of solutions to the regularized free boundary problem for the uniformly elliptic equation in the polar coordinates, and so does in the self-similar coordinates, as approximate solutions to the original free boundary problem. In §4, we proceed to the limits $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ to establish the global existence of solutions of the original problem in the self-similar coordinates. In §5, we establish the optimal $C^{0,1}$ -regularity of the solution ρ across the degenerate sonic boundary. In §6, we establish a corresponding theorem for the existence and regularity of solutions of the shock diffraction problem for the nonlinear wave system.

2. MATHEMATICAL FORMULATION AND MAIN THEOREM

In this section, we derive mathematical formulation of the shock diffraction problem as a free boundary problem for a nonlinear degenerate elliptic equation of second order and present our main theorem of this paper. In particular, we employ the Rankine-Hugoniot relations to set up a boundary condition along the free boundary (shock) and derive other boundary conditions along the wedge boundaries in the polar coordinates.

2.1. Rankine-Hugoniot Conditions and Oblique Derivative Boundary Condition on the Diffracted Shock. Consider system (1.8) in the polar coordinates. Then the Rankine-Hugoniot relations, i.e., the jump conditions, in the polar coordinates along the diffracted shock Γ_{shock} take the form:

$$(2.1) \quad -r[\rho] + \cos \theta[m] + \sin \theta[n] = \frac{dr}{d\theta} \left(-\frac{\sin \theta}{r}[m] + \frac{\cos \theta}{r}[n] \right),$$

$$(2.2) \quad \cos \theta[p] - r[m] = -\frac{dr}{d\theta} \frac{\sin \theta}{r}[p],$$

$$(2.3) \quad \sin \theta[p] - r[n] = \frac{dr}{d\theta} \frac{\cos \theta}{r}[p].$$

Solving for $[m]$ in (2.2) and for $[n]$ in (2.3), we have

$$(2.4) \quad [m] = \left(\frac{dr}{d\theta} \frac{\sin \theta}{r^2} + \frac{\cos \theta}{r} \right) [p], \quad [n] = \left(-\frac{dr}{d\theta} \frac{\cos \theta}{r^2} + \frac{\sin \theta}{r} \right) [p].$$

Combining (2.1) with (2.4), we find that, for our case,

$$(2.5) \quad \frac{dr}{d\theta} = r \frac{\sqrt{r^2 - \bar{c}^2(\rho, \rho_0)}}{\bar{c}(\rho, \rho_0)},$$

with $\bar{c}(\rho, \rho_0) = \sqrt{\frac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}$, where we have chosen the plus branch so that $\frac{dr}{d\theta} > 0$. Moreover, (2.1)–(2.4) imply that

$$(2.6) \quad [p][\rho] = [m]^2 + [n]^2.$$

Differentiating (2.6) along Γ_{shock} , i.e., $r'(\theta)\partial_r + \partial_\theta$, yields

$$(2.7) \quad (c^2[\rho] + [p])(\rho_r r'(\theta) + \rho_\theta) = 2[m](m_r r'(\theta) + m_\theta) + 2[n](n_r r'(\theta) + n_\theta).$$

From the second and third equations of (1.1) and $\omega = n_{x_1} - m_{x_2}$, we obtain

$$\omega_t = 0,$$

which means that, in any region formed by the flow trajectories across a curve along which the irrotational condition $\omega = 0$ holds, the classical solutions must be also irrotational. That is, $m_\eta = n_\xi$ holds in the region up to the shocks in the self-similar coordinates. Then, in such a region, using (1.8), we have

$$(2.8) \quad \begin{cases} m_r = \frac{c^2}{r} \cos \theta \rho_r - \frac{c^2}{r^2} \sin \theta \rho_\theta, \\ n_r = \frac{c^2}{r} \sin \theta \rho_r + \frac{c^2}{r^2} \cos \theta \rho_\theta, \\ m_\theta = (c^2 - r^2) \sin \theta \rho_r + \frac{c^2}{r} \cos \theta \rho_\theta, \\ n_\theta = -(c^2 - r^2) \cos \theta \rho_r + \frac{c^2}{r} \sin \theta \rho_\theta. \end{cases}$$

Hence, from (2.7), replacing (m_r, m_θ) and (n_r, n_θ) by the above equations and using $[m]$ and $[n]$ in (2.4), we obtain

$$(2.9) \quad \sum_{i=1}^2 \beta_i D_i \rho := \beta_1 \rho_r + \beta_2 \rho_\theta = 0,$$

where $\beta = (\beta_1, \beta_2)$ is a function of $(\rho_0, \rho, r(\theta), r'(\theta))$ with

$$(2.10) \quad \begin{aligned} \beta_1 &= r'(\theta)(c^2(r^2 - \bar{c}^2) - 3\bar{c}^2(c^2 - r^2)), \\ \beta_2 &= 3c^2(r^2 - \bar{c}^2) - \bar{c}^2(c^2 - r^2). \end{aligned}$$

Thus, the obliqueness becomes

$$0 \neq \beta \cdot (1, -r'(\theta)) = -2r^2(c^2 - \bar{c}^2)r'(\theta) =: \mu,$$

where $(1, -r'(\theta))$ is the outward normal to Ω on Γ_{shock} . Note that μ becomes zero when $r'(\theta) = 0$, that is, $r = \bar{c}(\rho, \rho_0)$. When the obliqueness fails, we have

$$\beta_1 = 0, \quad \beta_2 = -\bar{c}^2(c^2 - r^2) < 0,$$

since $c^2(\rho) > \bar{c}^2(\rho, \rho_0) = r^2$ if $\rho > \rho_0$.

We define Q to be the governing second-order quasilinear operator in the subsonic domain Ω :

$$(2.11) \quad Q\rho := ((c^2 - r^2)\rho_r)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0,$$

and M to be the oblique derivative boundary operator:

$$(2.12) \quad M\rho := \beta_1\rho_r + \beta_2\rho_\theta = 0 \quad \text{on } \Gamma_{\text{shock}} := \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\}.$$

The second condition on Γ_{shock} is the shock evolution equation:

$$(2.13) \quad \frac{dr}{d\theta} = r \frac{\sqrt{r^2 - \bar{c}^2(\rho, \rho_0)}}{\bar{c}(\rho, \rho_0)} := g(r, \theta, \rho(r, \theta)), \quad r(\theta_1) = r_1,$$

where (r_1, θ_1) are the polar coordinates of $P_1 = (\xi_1, \eta_1)$.

At the point P_2 , $r'(\theta_w) = 0$, M does not satisfy the oblique derivative boundary condition at this point. We may alternatively express this as a one-point Dirichlet condition by solving $r(\theta_w) = \bar{c}(\rho(r(\theta_w), \theta_w), \rho_0)$. In order to deal with this equation, we introduce the notation:

$$(2.14) \quad a = (\bar{c}_b)^{-1}(r) \quad \text{when } \bar{c}_b := \bar{c}(a, b) = r \text{ for fixed } b.$$

Thus, we have

$$(2.15) \quad \rho(P_2) = \bar{\rho} = (\bar{c}_{\rho_0})^{-1}(r(\theta_w)).$$

2.2. Boundary Condition on the Wedge. The boundary condition on the wedge is the slip boundary condition, i.e.,

$$(m, n) \cdot \nu = 0.$$

In order to derive the boundary condition for ρ itself, we notice that the normal direction along the wedge is $(-\eta, \xi)$ and the tangential direction is (ξ, η) , which implies that $\xi n - \eta m = 0$. Differentiating it along the wedge yields

$$0 = \xi^2 n_\xi - \xi \eta m_\xi + \xi n + \xi \eta n_\eta - \eta^2 m_\eta - \eta m = \xi^2 n_\xi - \xi \eta m_\xi + \xi \eta n_\eta - \eta^2 m_\eta.$$

Combining this with the second and third equations in (1.1), we conclude that ρ satisfies

$$(2.16) \quad \rho_\nu = 0 \quad \text{on } \Gamma_0 := \partial\Omega \cap (\{\theta = \pi\} \cup \{\theta = \theta_w\}).$$

2.3. Boundary Condition on Γ_{sonic} of State (1). The Dirichlet boundary condition on Γ_{sonic} :

$$(2.17) \quad \rho = \rho_1 \quad \text{on } \Gamma_{\text{sonic}} := \partial\Omega \cap \partial B_{c_1}(0).$$

On the Dirichlet boundary Γ_{sonic} , the equation $Q\rho = 0$ becomes degenerate elliptic from the inside of Ω .

2.4. Reformulation of the Shock Diffraction Problem. With the derivation of the free boundary condition on Γ_{shock} and the fixed boundary conditions on Γ_{sonic} and the wedge Γ_0 , *Problem 2* is reduced to the following free boundary problem in the domain Ω for the second order equation (2.11), with (m, n) correspondingly determined by (2.8).

Problem 3 (Free Boundary Problem). *Seek a solution of the second-order nonlinear equation (2.11) for the density function ρ in the domain Ω , satisfying the free boundary conditions (2.12)–(2.15) on Γ_{shock} , the Neumann boundary condition (2.16) on the wedge Γ_0 , and the Dirichlet boundary condition (2.17) on the degenerate boundary Γ_{sonic} (the sonic circle of state (1)) (see Fig. 2).*

2.5. Main Theorem. For the free boundary problem, *Problem 3*, we have the following results, which form the main theorem of this paper.

Theorem 2.1. (Main Theorem) *Let the wedge angle θ_w be between $-\pi$ and 0. Then there exists a global solution $\rho(r, \theta)$ in the domain Ω with the free boundary $r = r(\theta)$, $\theta \in [\theta_w, \theta_1]$, of Problem 3:*

$$\rho \in C^{2+\alpha}(\Omega) \cap C^\alpha(\overline{\Omega}), \quad r \in C^{2+\alpha}([\theta_w, \theta_1]) \cap C^{1,1}([\theta_w, \theta_1]).$$

Moreover, the solution $(\rho(r, \theta), r(\theta))$ satisfies the following properties:

- (i) $\rho > \rho_0$ on the shock Γ_{shock} , that is, the shock Γ_{shock} is separated from the sonic circle C_0 of state (0);
- (ii) The shock Γ_{shock} is strictly convex, except the point P_1 , in the self-similar coordinates (ξ, η) ;
- (iii) The solution is $C^{1,\alpha}$ up to Γ_{sonic} and Lipschitz continuous across Γ_{sonic} ;
- (iv) The Lipschitz regularity of solutions across Γ_{sonic} and at P_1 from the inside is optimal.

In particular, Theorem 2.1 implies the following facts:

- (i) The diffracted shock Γ_{shock} definitely is not degenerate at the point P_2 . This has been an open question even when the wedge angle is $\frac{\pi}{2}$ as in [18], though it is physically plausible.
- (ii) The curvature of the diffracted shock Γ_{shock} away from the point P_2 is strictly convex, though the strict convexity of the curvature fails at P_2 .
- (iii) The optimal regularity of solutions across Γ_{sonic} and at P_1 from the inside is $C^{0,1}$, i.e., Lipschitz continuity.

We establish Theorem 2.1 in two main steps. First, we solve the regularized approximate free boundary problem for Q involving two small parameters ε and δ , introduced in §3. Then we analyze the limits $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$, and prove that the limits yield a solution of *Problem 3*, i.e., (2.11)–(2.17), in §4. The optimal regularity is established in §5.

3. REGULARIZED APPROXIMATE PROBLEM

In this section we first formulate the regularized approximate free boundary problem and establish the existence of solutions to this problem as approximate solutions to the original problem.

To solve the free boundary problem, we formulate the fixed point argument in terms of the position of the free boundary. There are two main difficulties in establishing the existence of solutions: The first is that the ellipticity degenerates at the sonic circle Γ_{sonic} ; and the second is that the free boundary Γ_{shock} may coincide with the sonic circle C_0 of state (0) during iteration, which makes the iteration impossible. We overcome these difficulties as described below.

3.1. Approximate Problem and Existence Theorem for Approximate Solutions. For fixed ε , introduce a regularized operator

$$Q^\varepsilon := Q + \varepsilon \Delta,$$

where Δ represents the Laplace operator in the self-similar coordinates. For a given curve $r(\theta)$, we first solve the fixed boundary value problem (2.11)–(2.12), (2.16)–(2.17), and (2.15) with Q^ε replacing Q ; then we obtain a new shock position $\tilde{r}(\theta)$ by integrating (2.13):

$$(3.1) \quad \tilde{r}(\theta) = r_1 + \int_{\theta_1}^{\theta} g(r(s), s, \rho(s, r(s))) ds \quad \text{for } \theta \in [\theta_w, \theta_1],$$

where g is defined in (2.13). Note that, on the right side of (3.1), we evaluate all the quantities along the old shock position $r(\theta)$.

With this, it seems that the free boundary could be obtained by solving a fixed boundary problem and then by integrating the shock evolution equation. However, we face the second difficulty as indicated above, that is, $\tilde{r}(\theta)$ may meet the sonic circle C_0 of state (0). Introduce another small, positive parameter δ which is fixed and define the iteration set of r , $\mathcal{K}^{\varepsilon, \delta}$, which is a closed, convex subset of a Hölder space $C^{1+\alpha_1}([\theta_w, \theta_1])$, where α_1 depends on ε and δ to be specified later. The functions in $\mathcal{K}^{\varepsilon, \delta}$ satisfy

$$\begin{aligned} (K_1) \quad & r(\theta_1) = r_1; \\ (K_2) \quad & r'(\theta_w) = 0; \\ (K_3) \quad & c(\rho_0) + \delta \leq r(\theta_w); \\ (K_4) \quad & 0 \leq r'(\theta) \leq \frac{r_1^2}{c(\rho_0)} \quad \text{for } \theta_w \leq \theta \leq \theta_1. \end{aligned}$$

When the difficulty occurs, we modify $\tilde{r}(\theta)$ slightly somewhere as $r(\theta) = c(\rho_0) + \delta + A(\theta - \theta_w)^3 + B(\theta - \theta_w)^n$, where A, B , and n will be uniquely determined. Then we define a mapping on $\mathcal{K}^{\varepsilon, \delta}$:

$$J : r \rightarrow \tilde{r}.$$

We now restate the regularized approximate problem as follows: For fixed $\varepsilon, \delta > 0$, the equation for ρ in the subsonic region is

$$(3.2) \quad Q^\varepsilon \rho = ((c^2 - r^2 + \varepsilon)\rho_r)_r + \frac{c^2 + \varepsilon}{r} \rho_r + \left(\frac{c^2 + \varepsilon}{r^2} \rho_\theta\right)_\theta = 0;$$

the shock evolution equation remains the same when $r \geq c(\rho_0) + 2\delta$:

$$(3.3) \quad \begin{cases} \frac{dr}{d\theta} = g(r, \theta, \rho), \\ r(\theta_1) = r_1; \end{cases}$$

and

$$(3.4) \quad r(\theta) = c(\rho_0) + \delta + A(\theta - \theta_w)^3 + B(\theta - \theta_w)^n$$

for some constants A, B , and n on the boundary when (3.3) does not hold; the remaining boundary conditions as before are

$$(3.5) \quad M\rho = \beta \cdot \nabla \rho = 0 \quad \text{on } \Gamma_{\text{shock}} = \{(r, \theta) : \theta_w < \theta < \theta_1\},$$

$$(3.6) \quad \rho = \rho_1 \quad \text{on } \Gamma_{\text{sonic}}; \quad \rho_\nu = 0 \quad \text{on } \Gamma_0,$$

where ν is the outward normal to Ω at Γ_0 ; and

$$(3.7) \quad \rho(P_2) = \bar{\rho} = (\bar{c}_{\rho_0})^{-1}(r(\theta_w)).$$

Let $V = \{P_1, P_2, O, P_3\}$ denote the corners of Ω , and $V' = V \setminus \{P_2\}$. Set $\Omega' = \overline{\Omega} \setminus (V \cup \Gamma_{\text{shock}})$. For $P \in V$, we define the corner region

$$\Omega_P(\sigma) := \{x \in \Omega : \text{dist}(x, P) \leq \sigma\}, \quad \Omega_V(\sigma) := \cup_{P \in V} \Omega_P(\sigma).$$

We define a region that is close to Γ_{shock} , but does not contain the corner P_1 by taking a covering of Γ_{shock} with a ball of radius δ centered at the points on Γ_{shock} which are bounded away from P_1 . Define

$$\Gamma'(\sigma) := \{P \in \Gamma_{\text{shock}} : \text{dist}(P, P_1) > \sigma\}$$

and

$$\Gamma(\sigma) = \{x \in \Omega \cap (\cup_{P \in \Gamma'(\sigma)} B_\sigma(P))\},$$

where $B_\sigma(P)$ is a ball of radius σ centered at P . We then define

$$(3.8) \quad C_b^a \equiv \{u : \|u\|_a^b := \sup_{\sigma > 0} (\sigma^{a+b} \|u\|_{a, \bar{\Omega} \setminus (\Gamma(\sigma) \cup \Omega_{V'}(\sigma))}) < \infty\}.$$

We focus now on the proof of the following existence theorem in this section.

Theorem 3.1. *For any $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$ for some $\varepsilon_0, \delta_0 > 0$, there exists a solution $(\rho^{\varepsilon, \delta}, r^{\varepsilon, \delta}) \in C_{(-\gamma_1)}^{2+\alpha}(\Omega^{\varepsilon, \delta}) \times C^{1+\alpha}([\theta_w, \theta_1])$ to the regularized free boundary problem (3.2)–(3.7) such that*

$$(3.9) \quad \rho_0 < \bar{\rho}^{\varepsilon, \delta} \leq \rho^{\varepsilon, \delta} < \rho_1, \quad c^2(\rho^{\varepsilon, \delta}) \geq r^2 \quad \text{in } \overline{\Omega}^{\varepsilon, \delta}$$

for some $\alpha, \gamma \in (0, 1)$, which depend on ε, δ , and the data $(\rho_0, \rho_1, \theta_w)$. Furthermore, the solution satisfies (3.3) at the points of $\Gamma_{\text{shock}}^{\varepsilon, \delta}$ where $r^{\varepsilon, \delta} \geq c(\rho_0) + 2\delta$. The curve

$r^{\varepsilon, \delta}(\theta)$, defining the position of the free boundary $\Gamma_{shock}^{\varepsilon, \delta}$, is in $\mathcal{K}^{\varepsilon, \delta}$. Here $\Omega^{\varepsilon, \delta}$ is bounded by $\Gamma_{shock}^{\varepsilon, \delta}$, Γ_{sonic} , and Γ_0 .

We establish Theorem 3.1 in the following steps whose details are given in the following four subsections.

Step 1. Since the governing equation (3.2) is nonlinear and the ellipticity is not known a priori, we impose a cut-off function in the equation $Q^\varepsilon \rho = 0$.

We introduce a smooth increasing function $\zeta \in C^\infty$ such that

$$(3.10) \quad \zeta(s) := \begin{cases} s & \text{if } s \geq 0, \\ -\frac{1}{2}\varepsilon & \text{if } s < -\varepsilon, \end{cases}$$

and $|\zeta'(s)| \leq 1$. We then consider the following modified governing equation:

$$(3.11) \quad \begin{aligned} Q^{\varepsilon, +} \rho &= ((\zeta(c^2 - r^2) + \varepsilon)\rho_r)_r + (\frac{c^2 + \varepsilon}{r^2} \rho_\theta)_\theta + (\frac{1}{r}(\zeta(c^2 - r^2) + \varepsilon) + r)\rho_r \\ &= \sum_{i=1}^2 D_i(a_{ii}^\varepsilon(r, \theta, \rho)D_i \rho) + b^\varepsilon(r, \rho)D_r \rho = 0 \quad \text{in } \Omega. \end{aligned}$$

Step 2. We make some estimates for a solution to the linear problem with fixed boundary Γ_{shock} defined by $r(\theta) \in \mathcal{K}^{\varepsilon, \delta}$ and establish the Schauder estimates on Γ_{shock} .

For the linear problem, we establish a priori Schauder and Hölder bounds on Γ_{shock} , especially near the point where the boundary condition loses the obliqueness. We use the Hölder estimates of the neighborhood of the corners and the $C^{2+\alpha_-}$ estimates locally in the rest of the domain.

Step 3. We employ a technique in Lieberman [19] to solve the problem with the oblique derivative boundary condition $M\rho = 0$. Using the Hölder gradient bounds to the linear problem, we establish the existence results for the linear fixed boundary problem in the polar coordinates, via the Perron method developed in [19].

Step 4. We apply the Schauder fixed point theorem to conclude the existence of a solution to the free boundary problem with the oblique derivative boundary condition. Finally we remove the cut-off functions by the a priori estimate to conclude the results.

3.2. Proof of Theorem 3.1: Regularized Linear Fixed Boundary Value Problem. Replace ρ in the coefficients a_{ii}^ε and b^ε in (3.11) and β_i in (3.5) by a function w in a set \mathcal{W} that is defined with respect to a given boundary component $\Gamma_{shock}^{\varepsilon, \delta}$ and depends on the given values ρ_0 , ρ_1 , and $\bar{\rho}^{\varepsilon, \delta} = (\bar{c}_{\rho_0})^{-1}(r^{\varepsilon, \delta}(\theta_w))$:

Definition 3.2. The elements w of $\mathcal{W} \subset C_{(-\gamma_1)}^2$ satisfy

$$(W1) \quad \rho_0 < \bar{\rho}^{\varepsilon, \delta} \leq w \leq \rho_1, \quad w = \rho_1 \text{ on } \Gamma_{sonic}, \quad w_\nu = 0 \text{ on } \Gamma_0, \text{ and } w(P_2) = \bar{\rho}^{\varepsilon, \delta};$$

$$(W2) \quad \|w\|_{\alpha_0} \leq K_0, \quad \|w\|_{2+\alpha_0, \Omega_{loc}} \leq K_0, \text{ and } \|w\|_{1+\mu, \Gamma(d_0)} \leq K_0;$$

$$(W2) \quad \|w\|_2^{(-\gamma_1)} \leq K_1.$$

The weighted Hölder space is defined by (3.8). The values of $\gamma_1, \alpha_0, \mu \in (0, 1)$, and d_0 , as well as the values of K_0 and K_1 , will be specified later. Obviously, \mathcal{W} is closed, bounded, and convex.

The quasilinear equation (3.11) and the oblique derivative boundary condition (3.5) are now replaced by the linear equation and linear oblique derivative boundary condition on $\Gamma_{\text{shock}}^{\varepsilon, \delta} := \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\}$:

$$(3.12) \quad \begin{aligned} L^{\varepsilon, +} u &:= \sum_{i=1}^2 D_i(a_{ii}^{\varepsilon}(P, w) D_i u) + b^{\varepsilon}(P, w) D_1 u = 0 \quad \text{in } \Omega, \\ Mu &:= \beta_1(P, w) D_r u + \beta_2(P, w) D_{\theta} u = 0 \quad \text{on } \Gamma_{\text{shock}}^{\varepsilon, \delta}, \end{aligned}$$

with given $r(\theta) \in \mathcal{K}^{\varepsilon, \delta} \subset C^{1+\alpha_1}([\theta_w, \theta_1]) \cap C^2((\theta_w, \theta_1))$ and $w \in \mathcal{W}$, where the repeated indices are summed as usual. Because of the cut-off function ζ , $L^{\varepsilon, +}$ is uniformly elliptic in Ω with the ellipticity ratio depending on the data and ε .

In this section, we demonstrate the key point that, for a given function $w \in \mathcal{W}$, the solution u to the linear equation (3.12) with the remaining boundary conditions:

$$(3.13) \quad u = \rho_1 \quad \text{on } \Gamma_{\text{shock}}, \quad u_{\nu} = 0 \quad \text{on } \Gamma_0, \quad u(P_2) = \bar{\rho}^{\varepsilon, \delta},$$

satisfies the Hölder and Schauder estimates in Ω' and the uniform bound in $C^{1+\mu}(\Gamma(d_0))$ near $\Gamma_{\text{shock}}^{\varepsilon, \delta}$ for any $\mu < \min\{\gamma_1, \alpha_1\}$. This bound gives rise to enough compactness to establish the existence of a solution to the quasilinear problem by applying the Schauder fixed point theorem.

First, we state the Schauder estimates up to the fixed boundary Γ_{sonic} with the Dirichlet boundary condition, to Γ_0 with the Neumann boundary condition, and the Hölder estimates at the corners V' .

Lemma 3.3. *Assume that Γ_{shock} is parameterized as $\{(r(\theta), \theta)\}$ with $r(\theta) \in \mathcal{K}^{\varepsilon, \delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K_0, K_1, α_0 , and γ . Then there exist $\gamma_V, \alpha_{\Omega} \in (0, 1)$ such that any solution $u \in C_{\text{loc}}^{2+\alpha_{\Omega}}(\Omega') \cap C^{\gamma_V}(\Omega_{V'}(d_0))$ to the linear problem (3.12)–(3.13) satisfies*

$$(3.14) \quad \|u\|_{\gamma, \Omega_{V'}(d_0)} \leq C_1 \|u\|_0 \quad \text{for any } \gamma \leq \gamma_V,$$

and

$$(3.15) \quad \|u\|_{2+\alpha, \Omega'_{\text{loc}}} \leq C_2 \|u\|_0 \quad \text{for any } \alpha \leq \alpha_{\Omega}.$$

The exponent γ_V depends on the data ρ_0, ρ_1 , and θ_w ; and both α_{Ω} and γ_V depend on ε but are independent of α_1 and γ_1 . The constant C_2 is independent of K_1 but depends on K_0 .

Proof. The corner estimates at P_1 and P_3 directly follow from the results in Theorem 1, Lieberman [23]. Near the origin, the wedge angle is larger than π ; thanks to the symmetry of the governing equation in the θ -axis with form (1.13), we derive the corner estimate as follows.

First we flat out the boundary by introducing the transformation:

$$(r', \theta') = (r, \frac{\pi}{\pi - \theta_w}(\theta - \theta_w)), \quad (\xi', \eta') = (r' \cos \theta', r' \sin \theta').$$

Then the governing equation in the (r', θ') -coordinates takes the form

$$\tilde{Q}^{\varepsilon,+} \rho = ((\zeta(c^2(w) - r^2) + \varepsilon) \rho_r)_r + \left(\frac{\zeta(c^2(w) - r^2) + \varepsilon}{r} + r \right) \rho_r + \frac{\pi^2}{(\pi - \theta_w)^2} \left(\frac{c^2(w) + \varepsilon}{r^2} \rho_\theta \right)_\theta,$$

and

$$\begin{aligned} \tilde{Q}^{\varepsilon,+} \rho &= \left(((\zeta(c^2(w) - r^2) + \varepsilon) \frac{\xi^2}{r^2} + \frac{\pi^2}{(\pi - \theta_w)^2} \frac{(c^2 + \varepsilon) \eta^2}{r^2}) \rho_\xi \right)_\xi \\ &\quad + \left(((\zeta(c^2(w) - r^2) + \varepsilon) \frac{\xi \eta}{r^2} - \left(\frac{\pi^2}{(\pi - \theta_w)^2} \frac{(c^2 + \varepsilon) \xi \eta}{r^2} \right) \rho_\eta \right)_\xi \\ (3.16) \quad &\quad + \left(((\zeta(c^2(w) - r^2) + \varepsilon) \frac{\xi \eta}{r^2} - \frac{\pi^2}{(\pi - \theta_w)^2} \frac{(c^2 + \varepsilon) \xi \eta}{r^2}) \rho_\xi \right)_\eta \\ &\quad + \left(((\zeta(c^2(w) - r^2) + \varepsilon) \frac{\eta^2}{r^2} + \frac{\pi^2}{(\pi - \theta_w)^2} \frac{(c^2 + \varepsilon) \xi^2}{r^2}) \rho_\eta \right)_\eta + \xi \rho_\xi + \eta \rho_\eta \end{aligned}$$

in the (ξ', η') -coordinates, where we drop $'$ for simplicity without confusion. The eigenvalues of (3.16) are

$$\lambda_1 = \zeta(c^2 - r^2) + \varepsilon, \quad \lambda_2 = \left(\frac{\pi}{\pi - \theta_w} \right)^2 (c^2 + \varepsilon).$$

If we can show the transformation from the (ξ, η) -coordinates to (ξ', η') -coordinates is invertible and the C^α -norms are equivalent, then we can obtain the estimate of the solution at O by applying the standard Schauder-estimate since the coefficients of $\tilde{Q}^{\varepsilon,+} \rho$ are in L^∞ . In fact,

$$\begin{aligned} \frac{\partial \xi'}{\partial \xi} &= \cos \theta \cos \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right) + \frac{\pi}{\pi + \theta_0} \sin \theta \sin \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right), \\ \frac{\partial \xi'}{\partial \eta} &= \sin \theta \cos \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right) - \frac{\pi}{\pi + \theta_0} \cos \theta \sin \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right), \\ \frac{\partial \eta'}{\partial \xi} &= \cos \theta \sin \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right) - \frac{\pi}{\pi + \theta_0} \sin \theta \cos \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right), \\ \frac{\partial \eta'}{\partial \eta} &= \sin \theta \sin \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right) + \frac{\pi}{\pi + \theta_0} \cos \theta \cos \left(\frac{\pi}{\pi + \theta_0} (\theta + \theta_0) \right). \end{aligned}$$

Thus,

$$\det \left(\frac{D(\xi', \eta')}{D(\xi, \eta)} \right) \equiv \frac{\pi}{\pi - \theta_w} \quad \text{for all } (r, \theta) \in \mathbb{R}^2,$$

which implies that the transformation between the two coordinates is invertible.

As for the proof of the equivalence of the two norms, we have two cases:

Case 1. If $\theta \geq \frac{\pi}{2}$ as in Fig 3, then

$$\theta' = k\theta \geq \frac{\pi}{2} \quad \text{with } k = \frac{\pi - \theta_w}{\pi}.$$

Since $1 < k < 2$,

$$|x - y| \geq \max\{r(x), r(y)\}, \quad |x' - y'| \geq \max\{r(x'), r(y')\}.$$

Then the equivalence of the two $C^{1,1}$ -norms can be easily shown by setting $r(x) = r(x')$ and $r(y) = r(y')$.

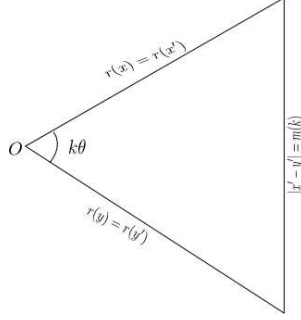


FIGURE 3. Scaling of the angles

Case 2. If $\theta < \frac{\pi}{2}$, then the distance between two points in the (ξ, η) -coordinates and (ξ', η') -coordinates is equivalent. By the cosine law, we define

$$m(k) := |x' - y'|^2 = r(x)^2 + r(y)^2 - 2r(x)r(y)\cos(k\theta),$$

and then

$$\frac{\partial m(k)}{\partial k} = 2kr(x)r(y)\sin(k\theta) > 0.$$

Thus,

$$|x - y| = \sqrt{m(1)} \leq \sqrt{m(k)} = |x' - y'| \leq \sqrt{m(2)} \leq 2\sqrt{m(1)} = 2|x - y|.$$

Therefore, we can obtain the Hölder estimate of the solution at O . Here γ_V depends on the angle at the corner, a fixed value that depends on the data $(\rho_0, \rho_1, \theta_w)$, and the ellipticity ratio ε , but independent of γ_1 , α_1 , K_0 , and K_1 .

Finally, we can use the standard interior and boundary Schauder estimates to obtain the local estimate (3.15). The constant C_2 depends on ε , the C^α -norm of the coefficients a_{ij} , and the domain. \square

Because the interior Schauder estimates can be further applied, a solution in $C_{\text{loc}}^{2+\alpha}(\Omega')$ is actually in $C_{\text{loc}}^3(\Omega)$.

We next establish the Hölder gradient estimates on Γ_{shock} . It is at this point that we need to derive the basic estimates at the point P_2 where the boundary operator M is not oblique. In order to avoid handling the Neumann boundary condition on the wedge boundary $\theta = \theta_w$ separately at each step of this proof, we reflect Ω across the wedge boundary $\theta = \theta_w$, without further comment, i.e., Ω includes Σ_0 , and let Γ_{shock} stand for the full $C^{1+\alpha_1}$ -boundary in Lemma 3.4 below. In addition, we extend $\tilde{u}(2\theta_w - \theta) = u(\theta)$ for $\theta \in (\theta_w, \theta_1)$ in a small neighborhood of θ_w . We still denote \tilde{u} by u for simplicity without confusion.

Lemma 3.4. *Assume that Γ_{shock} is given by $\{(r(\theta), \theta)\}$ with $r(\theta) \in \mathcal{K}^{\varepsilon, \delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K_0, K_1, α_0 , and γ_1 . Then there exists a positive constant d_0 such that, for any $d \leq d_0$, the solution $u \in C_{\text{loc}}^1(\Omega \cup \Gamma_{\text{shock}}) \cup C_{\text{loc}}^3(\Omega)$ to the linear problem (3.12)–(3.13) satisfies*

$$(3.17) \quad \|u\|_{1+\mu, \Gamma(d) \setminus B_d(P_1)} \leq C(\varepsilon, \delta, \alpha_1, \gamma_1, K_1, d_0) \|u\|_0$$

for any $\mu < \min\{\gamma_1, \alpha_1\}$.

The proof of this lemma is adopted from [5], which is long and tedious, so we postpone it in Appendix for self-contained.

Now we focus on the existence of solutions in Theorem 3.1 for problem (3.12)–(3.13).

First we introduce two definitions with some modification in comparison with [19].

We say that problem (3.12)–(3.13) is locally solvable if, for each $y \in \overline{\Omega}$, there exists a neighborhood $O(y)$ such that, for any $h \in C(\overline{N})$ with $N := O(y) \cap \{\overline{\Omega} \setminus (\{P_2\} \cup \Gamma_{\text{sonic}})\}$, there exists a solution $v \in C^2(N) \cap C(\overline{N})$ of the problem

$$\begin{aligned} L^{\varepsilon,+}v &= 0 && \text{in } N \cap \Omega, \\ Mv &= 0 && \text{on } N \cap \partial\Omega, \\ v &= h && \text{on } \partial'N, \end{aligned}$$

when $P_2 \notin N(y)$; or

$$\begin{aligned} L^{\varepsilon,+}v &= 0 && \text{in } N \cap \Omega, \\ Mv &= 0 && \text{on } N \cap \partial\Omega, \\ v &= h && \text{on } \partial'N, \\ v|_{P_2} &= \bar{\rho}^{\varepsilon,\delta}, \end{aligned}$$

when $P_2 \in N(y)$. Here $\partial'N = \partial N \cap \Omega$. For brevity, we denote this function v by $(h)_y$ to emphasize its dependence on h and y .

A subsolution (supersolution) of (3.12)–(3.13) is a function $w \in C(\overline{\Omega})$ with $w(r(\theta_w), \theta_w) \leq \bar{\rho}^{\varepsilon,\delta}$ ($w(r(\theta_w), \theta_w) \geq \bar{\rho}^{\varepsilon,\delta}$) such that, for any $y \in \overline{\Omega}$, if $h \geq w$ ($h \leq w$) on $\partial'N$, then $(h)_y \geq w$ ($(h)_y \leq w$) in N . The set of all subsolutions (supersolutions) is denoted by S^- (S^+).

We now establish the existence of solutions to problem (3.12)–(3.13).

Lemma 3.5. *Assume that Γ_{shock} is given by $\{(r(\theta), \theta)\}$ with $r(\theta) \in \mathcal{K}^{\varepsilon,\delta}$ for some α_1 and that $w \in \mathcal{W}$ for given K_0, K_1, α_0 , and γ_1 . Then there exist $\gamma_V, \alpha_\Omega \in (0, 1)$ and $d_0 > 0$, which are independent of γ_1 and α_1 , such that there exists a solution*

$$u^{\varepsilon,\delta} \in C^{1+\mu}(\Gamma(d) \setminus B_d(P_1)) \cap C^{2+\alpha}(\Omega') \cap C^\gamma(\Omega_{V'}(d))$$

to the linear problem (3.12)–(3.13) for any $\alpha \leq \alpha_\Omega$, $\mu < \min\{\gamma_1, \alpha_1\}$, $\gamma \leq \gamma_V$, and $d \leq d_0$, which satisfies (3.14)–(3.15) and (3.17).

Proof. For fixed $\varepsilon, \delta > 0$, we denote $u^{\varepsilon,\delta} = u$ in the proof without confusion. We use the Perron method to show the existence of a solution to problem (3.12)–(3.13).

It suffices to show the local existence at P_2 . In fact, let B_2 be a sufficiently small neighborhood of P_2 with smooth boundary such that $O \notin B_2$, $\beta_1 \leq 0$, and $\beta_2 < 0$. Then we study the local existence in the (ξ, η) -coordinates in B_2 . We introduce the coordinate transform in a neighborhood of P_2 :

$$(3.18) \quad \hat{\xi} = \hat{\xi}(r, \theta), \quad \hat{\eta} = \hat{\eta}(r, \theta)$$

such that

$$\begin{aligned}\hat{\xi}(r_w, \theta_w) &= 0, & \hat{\eta}(r_w, \theta_w) &= 0, \\ \frac{\partial \hat{\xi}}{\partial r} &= 0, & \frac{\partial \hat{\xi}}{\partial \theta} &= -\frac{1}{\beta_2} > 0, \\ \frac{\partial \hat{\eta}}{\partial r} &= -1, & \frac{\partial \hat{\eta}}{\partial \theta} &= -\frac{\beta_1}{\beta_2} \geq 0.\end{aligned}$$

Let $\Gamma_{shock} := \{(\hat{\xi}, \hat{\eta}) : \hat{\eta} = f(\hat{\xi})\} = \{(r, \theta) : r = r(\theta)\}$ in \hat{B}_2 . Then $\hat{\eta}(r(\theta), \theta) = f(\hat{\xi}(r(\theta), \theta))$ and hence

$$f'(\hat{\xi}) = \frac{\frac{\partial \hat{\eta}}{\partial r} r'(\theta) + \frac{\partial \hat{\eta}}{\partial \theta}}{\frac{\partial \hat{\xi}}{\partial r} r'(\theta) + \frac{\partial \hat{\xi}}{\partial \theta}} = -(\beta_1 - \beta_2 r'(\theta)) \geq 0,$$

and the function $f(\hat{\xi})$ is increasing in $\hat{\xi}$ on $\Gamma_{shock} \cap \hat{B}_2$. Thus, from $\frac{\partial \hat{\xi}}{\partial \theta} = -\frac{1}{\beta_2} > 0$ and $\frac{\partial \hat{\xi}}{\partial r} = 0$, we have

$$f(\hat{\xi}) \geq 0.$$

We reflect the region \hat{B}_2 across $\hat{\xi} = 0$ to obtain a new region, which is still denoted by \hat{B}_2 . Furthermore, we replace Ω by Ω_σ which is the σ -distance from the point P_2 upward, see Figure 4. On the bottom straight boundary of Ω_σ , we impose

$$u = \bar{\rho}^{\varepsilon, \delta} \quad \text{on bottom of } \Omega_\sigma.$$

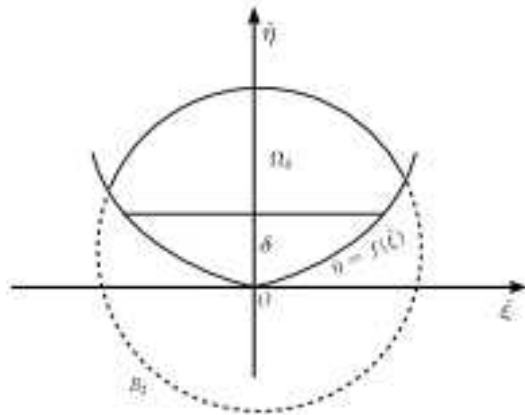


FIGURE 4. Domain with tip P_2 removed

Now we study the following boundary value problem:

$$(3.19) \quad \begin{cases} \hat{L}^{\varepsilon, \delta} u = \sum_{i,j=1}^2 \hat{a}_{ij} D_j u + \sum_{i=1}^2 \hat{b}_i D_i u = 0 & \text{in } \Omega_\sigma, \\ \hat{M} u = \partial_{\hat{\xi}} u = 0 & \text{on } \partial\Omega_\sigma \cap \Gamma_{\text{shock}}, \\ u = h & \text{on } \partial B_2 \cap \Omega, \\ u = \bar{\rho}^{\varepsilon, \delta} & \text{on } \Sigma_\sigma, \end{cases}$$

where

$$\begin{aligned} \tilde{a}_{11}^\varepsilon &= \frac{\hat{a}_{11}^\varepsilon}{\hat{\beta}_2^2}, & \tilde{a}_{12}^\varepsilon &= \tilde{a}_{21}^\varepsilon = -\frac{\hat{\beta}_1}{\hat{\beta}_2^2} \hat{a}_{22}^\varepsilon, & \tilde{a}_{22}^\varepsilon &= \hat{a}_{11}^\varepsilon + \left(\frac{\hat{\beta}_1}{\hat{\beta}_2}\right)^2 \hat{a}_{22}^\varepsilon, \\ \tilde{b}_1^\varepsilon &= \frac{\partial \hat{a}_{11}^\varepsilon}{\partial \hat{\eta}} - \frac{\hat{a}_{22}^\varepsilon}{\hat{\beta}_2^2} \frac{\partial \hat{\beta}_1}{\partial \hat{\xi}} + \frac{\hat{\beta}_1 \hat{a}_{22}^\varepsilon}{\hat{\beta}_2^2} \frac{\partial \hat{\beta}_2}{\partial \hat{\eta}} + \frac{\hat{\beta}_1 \hat{a}_{22}^\varepsilon}{\hat{\beta}_2^3} \frac{\partial \hat{\beta}_2}{\partial \hat{\xi}} - \frac{\hat{\beta}_1^2 \hat{a}_{22}^\varepsilon}{\hat{\beta}_2^3} \frac{\partial \hat{\beta}_2}{\partial \hat{\eta}} + \left(\frac{\hat{\beta}_1}{\hat{\beta}_2}\right)^2 \frac{\partial \hat{a}_{22}^\varepsilon}{\partial \hat{\eta}} - \hat{b}^\varepsilon, \\ \tilde{b}_2^\varepsilon &= -\frac{\hat{a}_{22}^\varepsilon}{\hat{\beta}_2^3} \frac{\partial \hat{\beta}_2}{\partial \hat{\xi}} + \frac{\hat{a}_{22}^\varepsilon \hat{\beta}_1}{\hat{\beta}_2^3} + \frac{1}{\hat{\beta}_2} \frac{\partial \hat{a}_{22}^\varepsilon}{\partial \hat{\xi}} - \frac{\hat{\beta}_1}{\hat{\beta}_2^2} \frac{\partial \hat{a}_{22}^\varepsilon}{\partial \hat{\eta}}. \end{aligned}$$

Here \hat{a}_{ii}^ε , \hat{b}^ε , and $\hat{\beta}_i$, $i = 1, 2$, are the coefficients of (3.12)–(3.13) in the $(\hat{\xi}, \hat{\eta})$ -coordinates, and h is a continuous function satisfying $\bar{\rho}^{\varepsilon, \delta} < h \leq \rho_1$. Following Lieberman [19], there exists a solution

$$u_\sigma \in C(\overline{\Omega} \cap \overline{\hat{B}_2}) \cup C^{2, \alpha}(\Omega_\sigma \cap \hat{B}_2)$$

for \hat{B}_2 small enough. The maximum principle holds for u_σ , which converges locally in $C^2(\Omega \cap \hat{B}_2)$ to a solution in $C^{2+ \alpha}(\Omega \cap \hat{B}_2)$ as $\sigma \rightarrow 0_+$.

We now use a barrier function to obtain the continuity of u at P_2 . We consider the auxiliary function

$$(3.20) \quad v = \bar{\rho}^{\varepsilon, \delta} + c(1 - e^{-l\hat{\eta}}),$$

where $c > 0$ and $l > 0$ are specified later. For the oblique derivative boundary condition along $\Omega_\sigma \cap \Gamma_{\text{shock}}$, we have the following two cases:

Case 1: $\tilde{\beta} \cdot \nu > 0$ and $\tilde{M}(v - \bar{\rho}^{\varepsilon, \delta}) \geq 0$ when $\hat{\xi} > 0$;

Case 2: $\tilde{\beta} \cdot \nu < 0$ and $\tilde{M}(v - \bar{\rho}^{\varepsilon, \delta}) \leq 0$ when $\hat{\xi} < 0$,

where ν denotes the outward normal to Ω_σ at $\Omega_\sigma \cap \Gamma_{\text{shock}}$.

Moreover, it is easy to see that $v \geq \bar{\rho}^{\varepsilon, \delta}$ on Σ_σ . Choose C large enough such that $v \geq \sup |h|$ on $\partial \hat{B}_2 \cap \Omega$. For the equation, we have

$$\tilde{a}_{ij}^\varepsilon D_j v + \tilde{b}_i^\varepsilon D_i v = -ce^{-l\hat{\eta}}(l^2 \tilde{a}_{11}^\varepsilon - l \tilde{b}_1^\varepsilon),$$

which is less than a negative constant by choosing $l > \|\tilde{b}^\varepsilon\|_0 / \lambda$, where $\lambda \leq \tilde{a}_{11}^\varepsilon(\hat{\xi}, \hat{\eta})$. Thus,

$$\bar{\rho}^{\varepsilon, \delta} \leq u \leq v.$$

Then u is continuous at the point P_2 . The continuity of u at the other points follows from Lieberman's argument in [21, 23].

By Lemma 3.4, we have $u \in C^{1, \mu}(\hat{B}_2 \cap \overline{\Omega})$.

In order to establish the global existence of solutions, it is required to show

$$\sup_{\Omega} (w^- - w^+) = 0,$$

where w^\pm are the supersolution and subsolution of problem (3.12)–(3.13), respectively.

In fact, we set $m := \sup_{\Omega} (w^- - w^+)$. We assume that $m > 0$ in Ω . Since $w^-(P_2) - w^+(P_2) \leq 0$, there exists a neighborhood $\hat{B}_2(P_2)$ of P_2 such that $w^-(y) - w^+(y) < m$ for $y \in \hat{B}_2(P_2)$. Now we define

$$\mathcal{Y} := \{y \in \overline{\Omega} : w^-(y) - w^+(y) = m\}.$$

Let $y_0 \in \mathcal{Y}$ such that

$$\text{dist}(y_0, P_2) = \min_{y \in \mathcal{Y}} \text{dist}(y, P_2).$$

Let \bar{w}^\pm be the lifts of w^\pm in $M(y_0)$. We see that

$$\bar{w}^- - \bar{w}^+ \leq m \quad \text{on } \partial' N.$$

The strong maximum principle implies that either $\bar{w}^- - \bar{w}^+ < m$ in M or $\bar{w}^- - \bar{w}^+ \equiv m$. Since $\bar{w}^-(y_0) - \bar{w}^+(y_0) \geq w^-(y_0) - w^+(y_0) = m$, it follows that

$$\bar{w}^- - \bar{w}^+ = m \quad \text{in } N,$$

and hence

$$\bar{w}^- - \bar{w}^+ \equiv m \quad \text{on } \partial' N,$$

which contains the point of \mathcal{Y} closer to P_2 than y_0 . This is a contradiction with the definition of y_0 .

We refer to Lieberman [21] to handle the mixed case and the points P_1 and P_3 , and Lieberman [22] to handle the point O where the two Neumann boundary conditions are satisfied. As for the interior and the Dirichlet boundary condition on the sonic arc Γ_{sonic} , they are classical since the equation is uniformly elliptic for fixed $\varepsilon > 0$ (see Gilbarg-Trudiger [16]).

With all of these, we then employ the Perron method to establish the existence of a global solution. \square

3.3. Proof of Theorem 3.1: Regularized Nonlinear Fixed Boundary Problem. We now establish the existence of solutions to the nonlinear problem (3.2) with a fixed boundary.

Lemma 3.6. *For $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, given $r(\theta) \in \mathcal{K}^{\varepsilon, \delta} \subset C^{1+\alpha_1}$, there exists a solution $\rho^{\varepsilon, \delta} \in C_{(-\gamma_1)}^{2+\alpha}(\Omega^{\varepsilon, \delta})$ to problem (3.2) and (3.6)–(3.7) with the oblique derivative condition $M\rho^{\varepsilon, \delta} = 0$ for some $\alpha(\varepsilon, \delta), \gamma_1(\varepsilon, \delta) \in (0, 1)$ such that*

$$(3.21) \quad \rho_0 < \bar{\rho}^{\varepsilon, \delta} \leq \rho^{\varepsilon, \delta} \leq \rho_1.$$

Moreover, for some $d_0 > 0$, the solution $\rho^{\varepsilon, \delta}$ satisfies

$$(3.22) \quad \|\rho^{\varepsilon, \delta}\|_{\gamma, \Gamma(d_0) \cup B_{d_0}(P_1)} \leq K_2,$$

where γ and K_2 depend on $\delta, \varepsilon, \gamma_V$, and K_1 , but are independent of α_1 .

Proof. For the notational simplicity, we write $\rho = \rho^{\varepsilon, \delta}$ throughout the proof.

For any function $w \in \mathcal{W}$, we define a mapping

$$T : \mathcal{W} \subset C_{(-\gamma_1)}^2 \rightarrow C_{(-\gamma_1)}^2$$

by $Tw = \rho$, where ρ is the solution to the linear regularized fixed boundary problem (3.12)–(3.13) solved in Lemma 3.5. Because of the cut-off function ζ , $L^{\varepsilon, +}$ is strictly elliptic, with the ellipticity ratio depending on ε . By Lemma 3.5, T maps \mathcal{W} into a bounded set in $C_{(-\gamma_V)}^{2+\alpha}$, where γ_V is the value given by Lemma 3.5. Since γ_V is independent of γ_1 , we may take $\gamma_1 = \gamma_V/2$ so that $T(\mathcal{K})$ is precompact in $C_{(-\gamma_1)}^2$.

To show that T maps \mathcal{W} into itself, we need to show that Tw satisfies (W1), (W2), and (W3) in Definition 3.2.

Now, (W1) is immediate by the boundary conditions and the maximum principle.

By applying the standard interior and boundary Hölder estimates (cf. Theorems 8.22 and 8.27 in [16]), we obtain

$$(3.23) \quad \|\rho\|_{\alpha^*, \Omega'_1} \leq C_0,$$

where $\alpha^* \in (0, 1)$ and C_0 depend only on ε (the ellipticity ratio), $d' = \text{dist}(\Omega'_1, \partial\Omega')$ with $\Omega'_1 \subset \Omega'$, and the data ρ_0 , ρ_1 , and θ_w . Notice that the constant C_0 is non-decreasing and the constant α^* is nonincreasing with respect to d' (see the remark following Theorem 8.24 in [16]). Since $\Omega' \subset \Omega$ is bounded, we can find an upper bound for C_0 and a lower bound for α^* depending only on the size of Ω and the ellipticity ratio. Thus, if we define \mathcal{W} with $K_0 = C_0$ and $\alpha_0 = \alpha^*$, then $\rho = Tw$ satisfies (W3). Note that K_0 and α^* are independent of α_1 and γ_V .

To verify (W2), it suffices to find a constant K such that

$$(3.24) \quad \sup_{\delta > 0} (\delta^{2-\gamma_1} \|\rho\|_{2, \bar{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}}) < K,$$

under the assumption that $\|w\|_2^{(-\gamma_1)} \leq K$. Note that Lemma 3.4 implies the existence of a constant $d_0 > 0$ such that, for $d \leq d_0$, any solution $u \in C^1(\Omega \cup \Gamma_{\text{shock}}) \cup C^3(\Omega)$ to the linear problem (3.12)–(3.13) satisfies (3.17), where the constant C depends on K but is uniform in $d \leq d_0$. Based on this estimate, we obtain a local bound for the weighted norm of ρ on $\Gamma(d_0)$ of the form

$$(3.25) \quad d^{2-\gamma_1} \|\rho\|_2 \leq d^{1-\gamma_1+\mu} C,$$

which holds for all $d < d_0$, where C depends on K , α_1 , and γ_1 . To show (3.24), we make the L^∞ -estimate by considering separately the domains in $\bar{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}$ for which $\delta > \tilde{d}$, with $\tilde{d} \leq d_0$ to be specified later, and the domains for which $\delta \leq \tilde{d}$.

In the domains of the first kind, $\bar{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}$ with $\delta > \tilde{d}$, the solution is smooth, and its C^2 -norm bound is independent of K . More precisely, we can use the uniform Hölder estimate (3.23) and the bootstrap iteratively to obtain

$$(3.26) \quad \|\rho\|_{2+\alpha_\Omega, \Omega'} \leq C(K_0).$$

Notice that, since the Hölder estimate (3.23) is independent of the distance between Ω'_1 and the boundary Γ_{shock} , so is the Schauder estimate (3.26). The interpolation

inequality (cf. Lemma 6.32, [16]) gives

$$(3.27) \quad \|\rho\|_{2,\Omega'} \leq c\|\rho\|_0 + \sigma\|\rho\|_{2+\alpha_\Omega,\Omega'} \leq c\rho_1 + \sigma C(K_0)$$

for any $\sigma > 0$ and $c = c(\sigma)$. We fix $\sigma = 1$ and find

$$(3.28) \quad \sup_{\delta > \tilde{d}} (\delta^{2-\gamma_1} \|\rho\|_{2,\overline{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}}) \leq K',$$

where K' depends on the size of the domain Ω , $C(K_0)$, and ρ_1 , but is independent of the distance to Γ_{shock} .

Next we estimate $\delta^{2-\gamma_1} \|\rho\|_{2,\overline{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}}$ with $\delta \leq \tilde{d}$. We divide the subdomain $\overline{\Omega} \setminus \{\Gamma(\delta) \cup \Omega_V(\delta)\}$ into two parts: The part for which $\delta > \tilde{d}$ and its complement. The supremum over the subdomain for which $\delta > \tilde{d}$ has been calculated above. We use the estimates for the behavior of the solution near Γ_{shock} to obtain the supremum over the complement, that is, estimate (3.25) and the corner estimate (3.14). In (3.14), the constants C_1 and γ_V are independent of K , K_0 , and α_1 , where $\|\rho\|_0$ is bounded by ρ_1 . By the interpolation inequality, we have

$$(3.29) \quad \|\rho\|_{\gamma_1,\Omega_V(d_V)} \leq C_V \|\rho\|_{\gamma_V,\Omega_V(d_V)} \leq C_V C_1 \rho_1,$$

since $\gamma_1 = \gamma_V/2$, where $C_V = C_V(\gamma_1, \gamma_V, \Omega_V(d_V))$ for some $d_V > 0$. From here, we obtain

$$d^{2-\gamma_1} \|\rho\|_2 \leq K_V, \quad \forall d < d_V,$$

where K_V is independent of K . Let $K = \max\{K_V, K'\}$. Using bound (3.28), K is independent of α_1 and \tilde{d} . Since K_V and K' are independent of \tilde{d} , we change \tilde{d} without affecting K . Therefore, we can choose $\tilde{d} \leq \min\{d_0, d_V\}/2$ in (3.25) small enough that

$$\tilde{d}^{1-\gamma_1+\mu} C \leq K.$$

Therefore, (3.24) is satisfied, and we have chosen the parameters K , K_0 , and α_0 defining \mathcal{W} so that T maps \mathcal{W} into itself.

Now, by the Schauder fixed point theorem, there exists a fixed point ρ such that $T\rho = \rho \in C_{(-\gamma_1)}^2$. Thus, ρ is a solution of the boundary value problem (3.2) and (3.5)–(3.7). By a bootstrap argument, we obtain

$$\rho \in C_{(-\gamma_1)}^{2+\alpha} \quad \text{for any } \alpha \leq \alpha_\Omega,$$

where α_1 is given in Lemma 3.3. Note that we have chosen $\gamma_1 = \gamma_V/2$ for the exponent $\gamma_V \in (0, 1)$ depending on ε and the corner angle α_Ω .

The maximum principle gives the first estimate in (3.21), and the second follows from Lemma 3.3.

Finally, since $T(\mathcal{W}) \subset \mathcal{W}$ is a bounded set in $C_{(-\gamma_1)}^2$, by the interpolation inequality, any fixed point ρ satisfies (3.22) for any $\gamma \leq \gamma_1$. Note that K_2 and γ_1 are independent of α_1 . \square

3.4. Proof of Theorem 3.1: Three Important Properties for the Nonlinear Problem. In this subsection, we show three properties for the fixed boundary nonlinear problem (3.2) and (3.5)–(3.7). We first prove the following inequality:

$$(3.30) \quad c^2(\rho^{\varepsilon,\delta}) - r^2 \geq 0 \quad \text{in } \overline{\Omega}^{\varepsilon,\delta},$$

which guarantees the ellipticity of our nonlinear system, so that we can remove the cut-off function.

Lemma 3.7. *There exist positive constants ε_0 and δ_0 such that, for any fixed $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta \leq \delta_0$, the solution $\rho^{\varepsilon,\delta} \in C(\overline{\Omega}) \cap C^2(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{sonic}})$ to (3.2)–(3.7) satisfies*

$$(3.31) \quad c^2(\rho^{\varepsilon,\delta}) \geq r^2 \quad \text{in } \overline{\Omega}^{\varepsilon,\delta}.$$

Proof. For the notational simplicity, throughout the proof, we write $\rho = \rho^{\varepsilon,\delta}$.

We prove the result by contradiction arguments. On contrary, we assume that there exists a nonempty set $D = \{(\xi, \eta) \in \overline{\Omega} : c^2(\rho) - r^2 < 0\}$, and choose $X_{\min} \in D$ to be the minimum point. Then it is easy to check that $P_2 \notin D$. Since $O \notin D$,

$$D \subset \Omega_s := \{X \in \overline{\Omega} \setminus V : r^2 > c^2(\rho, \rho_0)\},$$

where V is the set of the corner points of Ω . Thus, there are three possible locations of X_{\min} .

Case 1: $X_{\min} \in D$ locates inside Ω_s . For notational simplicity, we denote $c^2(\rho) = \rho^{\gamma-1} = u$. Then, multiplying $(\gamma - 1)\rho^{\gamma-2}$ both sides of the equation $Q^{\varepsilon,+}\rho = 0$, we have

$$(3.32) \quad \begin{aligned} Lu &= (\gamma - 1)\rho^{\gamma-2}Q^{\varepsilon,+}\rho \\ &= \sum_{i=1}^2 a_{ii}^{\varepsilon}(D_{ii}u - \frac{\gamma-2}{\gamma-1}\frac{1}{\rho^{\gamma-1}}|D_i u|^2) + \zeta'(c^2 - r^2)(c^2 - r^2)_r u_r + \frac{1}{r^2}u_{\theta}^2 + b^{\varepsilon}u_r \\ &= 0. \end{aligned}$$

We note that $\frac{\varepsilon}{2} \leq a_{11}^{\varepsilon} \leq \varepsilon$ due to the cut-off function ζ in D . We evaluate Lr^2 in D :

$$(3.33) \quad \begin{aligned} Lr^2 &\geq -2\varepsilon \left| 1 - \frac{2(\gamma-2)}{\gamma-1}\frac{1}{\rho^{\gamma-1}}r^2 \right| + \zeta'(c^2 - r^2)(c^2 - r^2)_r u_r + 2c^2 \\ &\geq 2r_0^2 - \frac{2\varepsilon}{\rho_0^{\gamma-1}} \left| \rho_0^{\gamma} - 1 - \frac{2|\gamma-2|}{\gamma-1}r_0^2 \right| > 0 \end{aligned}$$

with small $\varepsilon < \varepsilon_0$, where

$$\varepsilon = \frac{\rho_0^{\gamma-1}r_0^2}{\left| \rho_0^{\gamma-1} - \frac{2|\gamma-2|}{\gamma-1}r_0^2 \right|}$$

and the fact that $(c^2 - r^2)_r(X_{\min}) = 0$. Then, in $D \cap \Omega$, we obtain

$$(3.34) \quad \begin{aligned} 0 &> Lu - Lr^2 \\ &= \sum_{i=1}^2 \left(a_{ii}^{\varepsilon}D_{ii}(u - r^2) - \frac{\gamma-2}{(\gamma-1)\rho^{\gamma-1}}a_{ii}^{\varepsilon}D_i(u + r^2)D_i(u - r^2) \right) \\ &\quad + \zeta'(c^2 - r^2)(c^2 - r^2)_r(u - r^2)_r + \frac{1}{r^2}(u + r^2)_{\theta}(u - r^2)_{\theta} + b(u - r^2)_r. \end{aligned}$$

Since X_{\min} is an interior minimum point,

$$D_i(u - r^2)(X_{\min}) = 0, \quad \sum_{i=1}^2 a_{ii}^\varepsilon D_{ii}(u - r^2)(X_{\min}) \geq 0,$$

which contradicts the inequality $Lu - Lr^2 < 0$ in $D \cap \Omega$.

Case 2: X_{\min} locates on $\Gamma_{\text{shock}} \cap D$. Multiplying $(\gamma - 1)\rho^{\gamma-2}$ over the equation $M\rho = 0$, we have

$$0 = (\gamma - 1)\rho^{\gamma-2}M\rho = \tilde{M}u = \sum_{i=1}^2 \beta_i D_i u.$$

On one hand, we have

$$(3.35) \quad \tilde{M}r^2 = 2r\beta_1 = 2rr'(c^2(r^2 - \bar{c}^2) - 3\bar{c}^2(c^2 - r^2)) > 0 \quad \text{on } \Gamma_{\text{shock}} \cap D,$$

where we have used the fact that $r^2 \geq c^2 \geq \bar{c}^2$ in Ω_s .

On the other hand, at X_{\min} , the outward normal derivative of $u - r^2$ becomes non-positive (i.e., $\nabla(u - r^2)(1, -r') \leq 0$) and the tangential derivative becomes zero (i.e., $\nabla(u - r^2)(r', 1) = 0$). We obtain

$$(1 + (r')^2)(u - r^2)_r \leq 0 \quad \text{at } X_{\min},$$

which implies $(u - r^2)_r \leq 0$. Thus we have

$$0 > \tilde{M}(u - r^2) = (\beta_1 - r'\beta_2)(u - r^2)_r = \mu(u - r^2)_r \geq 0,$$

which is a contradiction.

Case 3: X_{\min} locates on $\Gamma_0 \cap D$. Then

$$(\gamma - 1)\rho^{\gamma-2} \frac{\partial \rho}{\partial \boldsymbol{\nu}} - \frac{\partial r^2}{\partial \boldsymbol{\nu}} = 0,$$

which is a contradiction due to the Hopf maximum principle, i.e.,

$$\frac{\partial(u - r^2)}{\partial \boldsymbol{\nu}}(X_{\min}) < 0.$$

Therefore, there is no minimum point, which implies the set $D = \emptyset$. This completes the proof. \square

Based on Lemma 3.7, we can show that the solution of the free boundary problem (3.2)–(3.7) satisfies

$$r - \bar{c}(\rho, \rho_0) \geq 0 \quad \text{on } \Gamma_{\text{shock}}.$$

Lemma 3.8. *Let $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta \leq \delta_0$, and let*

$$\rho^{\varepsilon, \delta} \in C(\bar{\Omega}) \cap C^2(\Omega) \cap C^1(\Omega \setminus \Gamma_{\text{sonic}})$$

be a solution of the boundary value problem (3.2) and (3.5)–(3.7). Then

$$\bar{c}(\rho^{\varepsilon, \delta}, \rho_0) - r \leq 0 \quad \text{on } \Gamma_{\text{shock}}^{\varepsilon, \delta}.$$

The basic idea of the proof of this lemma is from [18]. For self-containedness, we include the proof in Appendix.

Based on Lemma 3.8, it is easy to show that $r(\theta) > \bar{c}(\rho^{\varepsilon,\delta}(r(\theta), \theta), \rho_0)$ on $\Gamma_{\text{shock}}^{\varepsilon,\delta}$ for $\theta \in (\theta_w, \theta_1)$. Thus, we have shown that the integration in (3.1) can always be integrated on Γ_{shock} .

In the next lemma, we study the monotonicity of ρ along the shock boundary Γ_{shock} , which will be used to describe the behavior of $\rho^{\varepsilon,\delta}$ and $r^{\varepsilon,\delta}$ near the shock $\Gamma_{\text{shock}}^{\varepsilon,\delta}$ when ε, δ tend to zero, and the convexity of the shock in the (ξ, η) -coordinates.

Lemma 3.9. *Suppose that $\rho^{\varepsilon,\delta} \in C^1(\Omega \cup \Gamma_{\text{shock}} \cup \Gamma_{\text{wedge}} \cup \Sigma_0) \cap C^\alpha(\bar{\Omega})$ is a solution of the boundary value problem (3.2) and (3.5)–(3.7). Then $\rho^{\varepsilon,\delta}$ is monotone on Γ_{shock} .*

The idea of the proof of this lemma is from [5], and the main difference here is that we only need the uniform C^α -regularity. We postpone this proof in Appendix for self-containedness.

3.5. Proof of Theorem 3.1: The Regularized Nonlinear Free Boundary Problem. We now show the existence of a solution to the regularized free boundary problem.

Lemma 3.10. *For each $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$ with some $\varepsilon_0 > 0$ and $\delta_0 > 0$, there exists a solution $(\rho^{\varepsilon,\delta}, r^{\varepsilon,\delta}) \in C_{(-\gamma)}^{2+\alpha}(\Omega^{\varepsilon,\delta}) \times C^{1+\alpha_1}([-\frac{\pi}{2}, \theta_1])$ to the regularized free boundary problem (3.2)–(3.7).*

Proof. For the notational simplicity, we suppress the (ε, δ) -dependence in the proof.

For each $r(\theta) \in \mathcal{K}^{\varepsilon,\delta} \subset C^{1+\alpha_1}([\theta_w, \theta_1])$, using the solution ρ of the nonlinear fixed boundary problem (3.2) and (3.6)–(3.7) given by Lemma 3.5, we define the map J on \mathcal{K} , $\tilde{r} = Jr$, as in (3.1):

$$(3.36) \quad \tilde{r}(\theta) = r_1 + \int_{\theta_1}^{\theta} g(r(s), s, \rho(r(s), s)) ds.$$

There are two cases for the approximate shock position $\tilde{r}(\theta)$:

Case 1: $\tilde{r}(\theta_w) \geq c(\rho_0) + \delta$. We check that J maps \mathcal{K} into itself. It is easy to check that $\tilde{r}(\theta) \in C^{1+\gamma\nu}([\theta_w, \theta_1]) \cap C^{1+1}([\theta_w, \theta_1])$, from the definition of \tilde{r} and by using Lemmas 3.3–3.4. property (K_1) follows from (3.36). By the definition of g and $\rho(P_2) = \bar{\rho}$, $\tilde{r}'(\theta) = 0$ holds, which implies property (K_2) . Then it suffices to show that property (K_4) holds, since the upper and lower bounds of ρ , Lemma 3.8, and (K_4) imply (K_3) . From the expression of $g(r(\theta), \theta, \rho(r(\theta), \theta))$ and the upper and lower bounds of ρ , we have (K_4) .

Case 2: $\tilde{r}(\theta_w) < c(\rho_0) + \delta$. Since $\tilde{r}'(\theta) > 0$ for $\theta \in (\theta_w, \theta_1)$ and $r_1 = c(\rho_1) > c(\rho_0) + \delta$, there exists a unique $\theta_a \in (\theta_w, \theta_1)$ such that $\tilde{r}(\theta_a) = c(\rho_0) + \delta$. Now, choosing τ to be determined later such that $\tilde{r}(\theta_a + \tau) \leq c(\rho_0) + 2\delta$ and letting $x_1 = \theta_a + \tau - \theta_w$, we modify the approximate shock position on $\theta_w \leq \theta \leq \theta_a + \tau$ by defining

$$\hat{r}(\theta) = c(\rho_0) + \delta + A(\theta - \theta_w)^3 + B(\theta - \theta_w)^n$$

with

$$A = \frac{1}{(n-3)x_1^3}(na - bx_1), \quad B = \frac{1}{(n-3)x_1^n}(bx_1 - 3a),$$

where $a = \tilde{r}(\theta_a + \tau) - c(\rho_0) - \delta$ and $b = \tilde{r}'(\theta_a + \tau)$.

Choose τ small enough such that

$$bx_1 - 3a > 0,$$

and then n sufficiently large such that

$$na - bx_1 > 0,$$

where n depends on δ , but independent of the iteration. Next, we choose n and τ precisely. In fact, it is easy to see that

$$|b| \leq \frac{c(\rho_1)}{c(\rho_0)} \sqrt{c^2(\rho_1) - c^2(\rho_0)},$$

and

$$|bx_1| \leq C(\rho_0, \rho_1, \theta_1, \theta_w) := \frac{c(\rho_1)}{c(\rho_0)} \sqrt{c^2(\rho_1) - c^2(\rho_0)} (\theta_1 - \theta_w).$$

If $3\delta \leq bx_1$, we choose τ such that $a = \delta$ and $n_1 = \frac{C(\rho_0, \rho_1, \theta_1, \theta_w)}{\delta} + 1$, which depend only on δ , ρ_0 , and ρ_1 .

If $3\delta > bx_1$, letting τ small enough, we can obtain new a and b satisfying $3a = bx_1$, where we choose the biggest τ smaller than the old one such that $3a = bx_1$ holds. Note that $bx_1 > 0$ and $\tilde{r}(\theta_a) = c(\rho_0) + \delta$. Thus, choosing $n_2 = 4$, we have $A > 0$ and $B = 0$.

Let $n = \max(n_1, n_2) = n(\rho_0, \rho_1, \theta_1, \theta_w, \delta)$, which is independent of the iteration process. Thus, $\hat{r}(\theta)$, uniquely determined, is a strictly increasing function on $[\theta_w, \theta_a + \tau]$. Furthermore, we have

$$0 = \hat{r}'(-\frac{\pi}{2}) \leq \hat{r}'(\theta) \leq \hat{r}'(\theta_a + \tau) = \tilde{r}'(\theta_a + \tau).$$

We define

$$Jr(\theta) = \begin{cases} \tilde{r}(\theta) & \text{for } \theta \in [\theta_a + \tau, \theta_1], \\ \hat{r}(\theta) & \text{for } \theta \in [\theta_w, \theta_a + \tau]. \end{cases}$$

It is easy to show that $Jr(\theta)$, $\theta \in [\theta_w, \theta_1]$, satisfies properties (K_1) – (K_4) .

First, since $\tilde{r}(\theta) \in C^{1+\gamma_V}([\theta_a + \tau, \theta_1])$, $\hat{r}(\theta) \in C^\infty([\theta_w, \theta_a + \tau])$, and $(Jr)'(\theta) \in C([\theta_w, \theta_1])$, we have

$$Jr(\theta) \in C^{1+\gamma_V}([\theta_w, \theta_1]).$$

Next, for $\theta \in [\theta_w, \theta_a + \tau]$, $\hat{r}'(\theta) = 3A(\theta - \theta_w)^2 + nB(\theta - \theta_w)^{n-1}$. Then

$$\begin{aligned} & \hat{r}'(\theta_2) - \hat{r}'(\theta_3) \\ &= 3A(\theta_2 - \theta_w)^2 - 3A(\theta_3 - \theta_w)^2 + nB(\theta_2 - \theta_w)^{n-1} - nB(\theta_3 - \theta_w)^{n-1} \\ &= 3A(\theta_2 - \theta_3)(\theta_2 + \theta_3 - 2\theta_w) + nB(\theta_2 - \theta_3) \left(\sum_{j=0}^{n-2} C_{n-2}^j (\theta_2 - \theta_w)^{n-2-j} (\theta_3 - \theta_w)^j \right). \end{aligned}$$

Using the fact that $\theta_2 - \theta_w, \theta_3 - \theta_w \leq x_1$, and $A, B \geq 0$, we obtain

$$\begin{aligned} |\hat{r}(\theta_2)' - \hat{r}(\theta_3)'| &\leq |\theta_2 - \theta_3|^\alpha (6Ax_1^{2-\alpha} + C(n)Bx_1^{n-1-\alpha}) \\ &\leq C(n)(ax_1^{-1-\alpha} + bx_1^{-\alpha})|\theta_2 - \theta_3|^\alpha. \end{aligned}$$

Notice that $\tilde{r}' = \frac{r}{c}\sqrt{r^2 - \bar{c}^2}$, $r \in C^{1+\gamma_V}$, and $\theta_a + \tau$ is uniformly away from θ_1 , which means $\rho \in C^{1+\mu}$. We obtain

$$\tilde{r}' \leq C(\rho_0, \rho_1, \varepsilon, \delta)x_1^{1/2},$$

which implies

$$|\hat{r}'(\theta_2) - \hat{r}'(\theta_3)| \leq C(\rho_0, \rho_1, \varepsilon, \delta)|\theta_2 - \theta_3|^\alpha \quad \text{if } \alpha \leq \frac{1}{2}.$$

Thus

$$\|Jr\|_{C^{1+\alpha}([-\frac{\pi}{2}, \theta_1])} \leq C(\rho_1, \rho_2, \varepsilon, \delta)$$

if $\alpha \leq \min\{\gamma_V, \frac{1}{2}\}$, which satisfies $(K_1)-(K_4)$.

Thus, we define a map

$$J : \mathcal{K}^{\varepsilon, \delta} \rightarrow \mathcal{K}^{\varepsilon, \delta}$$

by

$$\tilde{r} = Jr.$$

Obviously, $\mathcal{K}^{\varepsilon, \delta}$ is a convex and closed subset of the Banach space C^{α_1} , and J is compact, if $\alpha_1 < \min\{\gamma_V, \frac{1}{2}\}$. In order to use the Schauder fixed point theorem, we need to prove that J is continuous on $\mathcal{K}^{\varepsilon, \delta}$.

Assume that $r_m, r \in \mathcal{K}^{\varepsilon, \delta}$ for $m = 1, 2, \dots$, $r_m \rightarrow r$ as $m \rightarrow \infty$, and ρ_m solves the fixed boundary problem for r_m for each fixed m . Then, by the standard argument as in [5], $\rho_m \rightarrow \rho$, which solves the problem for r . Therefore, we have

$$g(r_m(\theta), \theta, \rho_m(r_m(\theta), \theta)) \longrightarrow (r(\theta), \theta, \rho(r(\theta), \theta)) \quad m \rightarrow \infty,$$

which implies $Jr_m \rightarrow Jr$ as $m \rightarrow \infty$ at the point where (3.3) holds for both r_m and r . Then $Jr_m \rightarrow Jr$ as $m \rightarrow \infty$, if Jr belongs to Case 1. For Case 2, due to the construction, we divide it into three subcases:

$$3\delta < bx_1; \quad 3\delta > bx_1; \quad 3\delta = bx_1,$$

where $b = \tilde{r}'(\theta_a + \tau)$, $x_1 = \theta_a + \tau - \theta_w$, and $\tilde{r}(\theta_a + \tau) = c(\rho_0) + 2\delta$ depend only on r and δ . For any case, it is easy to deduce that

$$(\tau_m, \theta_{a,m}) \rightarrow (\tau, \theta_a), \quad (A_m, B_m) \rightarrow (A, B) \quad m \rightarrow \infty.$$

Then $Jr_m \rightarrow Jr$, with the fact that

$$Jr_m = c(\rho_0) + \delta + A_m(\theta - \theta_w)^3 + B_m(\theta - \theta_w)^n$$

for $\theta < \theta_{a,m} + \tau_m$, where n , θ_w , and ρ_0 are universal constants.

Then, for any fixed $\varepsilon, \delta > 0$, we obtain the existence of a solution $(\rho^{\varepsilon, \delta}, r^{\varepsilon, \delta})$ to the free boundary problem by the standard fixed point argument. Moreover, we have $r^{\varepsilon, \delta} \in C^{1+\alpha}([\theta_w, \theta_1])$ for $\alpha \leq \alpha_1$. This completes the proof. \square

3.6. Proof of Theorem 3.1: Completion. We note that Lemma 3.10 implies that there exists a solution $(\rho^{\varepsilon,\delta}, r^{\varepsilon,\delta})$ such that $r^{\varepsilon,\delta} \in \mathcal{K}^{\varepsilon,\delta}$. From Lemma 3.6 and the interior Schauder estimate, we note that $\|\rho^{\varepsilon,\delta}\|_{C_{loc}^{2,\alpha}} \leq C$, and $\rho^{\varepsilon,\delta}$ satisfies property (3.9). By Lemma 3.7, we have $c^2(\rho^{\varepsilon,\delta}) \geq r^2$. This completes the proof of Theorem 3.1.

4. PROOF OF THEOREM 2.1: EXISTENCE OF SOLUTIONS

In this section, we study the limiting solution, as the elliptic regularization parameter ε and the oblique derivative boundary regularization parameter δ tend to 0. We start with the regularized solutions of problem (3.2) and (3.5)–(3.7), whose existence is guaranteed by Theorem 3.1. Denote by $\rho^{\varepsilon,\delta}$ a sequence of the regularized solutions of the boundary value problem.

We first construct a uniform lower barrier to obtain the uniform ellipticity in any compact domain contained by $\bar{\Omega} \setminus (\Gamma_{\text{sonic}} \cup \Gamma'_{\text{sonic}})$ for the solutions of the regularized problems.

Lemma 4.1. *There exists a positive function φ , independent of ε and δ , such that*

$$\varphi \rightarrow 0 \quad \text{as} \quad \text{dist}((\xi, \eta), \Gamma_{\text{sonic}} \cup \Gamma'_{\text{sonic}}) \rightarrow 0,$$

and

$$c^2(\rho^{\varepsilon,\delta}) - (\xi^2 + \eta^2) \geq \varphi \quad \text{in} \quad \bar{\Omega} \setminus (\Gamma_{\text{sonic}} \cup \Gamma'_{\text{sonic}}).$$

Proof. For the notational simplicity, throughout the proof, we write $\rho = \rho^{\varepsilon,\delta}$.

1. We claim that $c^2(\rho) - \bar{c}^2(\rho, \rho_0)$ is increasing as a function of ρ for $\rho > \bar{\rho} > \rho_0$. In fact, we set

$$g(\rho; \rho_0) = c^2(\rho) - \bar{c}^2(\rho, \rho_0) \quad \text{for } \rho \in [\bar{\rho}, \rho_1].$$

Then

$$g'(\rho; \rho_0) = \frac{p''(\rho)(\rho - \rho_0)^2 - p'(\rho)(\rho - \rho_0) + p(\rho) - p(\rho_0)}{(\rho - \rho_0)^2}.$$

In order to show $g'(\rho; \rho_0) > 0$ for $\rho > \bar{\rho} > \rho_0$, we define

$$h(\rho; \rho_0) = p''(\rho)(\rho - \rho_0)^2 - p'(\rho)(\rho - \rho_0) + p(\rho) - p(\rho_0),$$

which is the denominator of $f'(\rho)$. Then

$$\begin{aligned} (4.1) \quad h'(\rho; \rho_0) &= p'''(\rho)(\rho - \rho_0)^2 + p''(\rho)(\rho - \rho_0) \\ &= (\rho - \rho_0)((\gamma - 1)(\gamma - 2)\rho^{\gamma-3}(\rho - \rho_0) + (\gamma - 1)\rho^{\gamma-2}) \\ &= (\gamma - 1)(\rho - \rho_0)\rho^{\gamma-3}((\gamma - 1)\rho - (\gamma - 2)\rho_0) > 0 \quad \text{for } \rho > \rho_1 > \rho_0, \end{aligned}$$

where we have used the fact that $c^2(\rho) = p'(\rho) = \rho^{\gamma-1}$. Thus, $h(\rho; \rho_0) > 0$ since $h(\rho_0) = 0$, which implies that $g(\rho; \rho_0) > 0$ for $\rho > \bar{\rho} > \rho_0$.

2. Since $\rho \geq \bar{\rho}$ in $\bar{\Omega}$, we have $c^2(\rho) \geq c^2(\bar{\rho}) > \bar{c}^2(\bar{\rho}, \rho_0)$. Let δ_1 be a positive constant:

$$\delta_1 := c^2(\bar{\rho}) - \bar{c}^2(\bar{\rho}, \rho_0) < c^2(\rho) - \bar{c}^2(\rho, \rho_0).$$

Hence, ρ satisfies $c^2(\rho) - (\xi^2 + \eta^2) \geq \delta_1$ for $r = \sqrt{\xi^2 + \eta^2} < \frac{c(\rho_0)}{2}$ in Ω . It follows from (3.21) in Lemma 3.6 that there exists a constant $e_0 > 0$ independent of ε such that $\delta_1 \geq e_0 > 0$. From now on, we only consider the set:

$$\tilde{\Omega} = \{P \in \bar{\Omega} \setminus \Gamma_{\text{sonic}} : r \geq \frac{c(\rho_0)}{2}\}.$$

For $0 < R < 1$ and $X_0 = (\xi_0, \eta_0) \in \tilde{\Omega}$, let

$$\zeta(X) = 1 - \frac{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}{R^2} \quad \text{for } B_R(X_0) \cap \Gamma_{\text{sonic}} = \emptyset.$$

We define

$$\varphi(\xi, \eta) = \delta_0 (\zeta(X))^\tau,$$

where δ_0 and τ are two positive constants to be specified later, and $0 \leq \zeta \leq 1$ independent of ε .

3. Multiplying the equation $Q\rho + \varepsilon \Delta \rho = 0$ by $(\gamma - 1)\rho^{\gamma-2}$ and letting $c^2(\rho) =: u$, we have

$$(4.2) \quad Lu = \sum_{i,j=1}^2 a_{ij} D_{ij}u + \sum_{i=1}^2 a_i (D_i u)^2 + a_3 \sum_{i,j=1}^2 D_i u D_j u + \sum_{i=1}^2 b_i D_i u = 0,$$

where

$$\begin{aligned} a_{11} &= c^2(\rho) - \xi^2 + \varepsilon, & a_{22} &= c^2(\rho) - \eta^2 + \varepsilon, & a_{12} &= a_{21} = -\xi\eta, \\ a_1 &= 1 - \frac{\gamma-2}{(\gamma-1)c^2(\rho)}(c^2(\rho) - \xi^2 + \varepsilon), & a_2 &= 1 - \frac{\gamma-2}{(\gamma-1)c^2(\rho)}(c^2(\rho) - \eta^2 + \varepsilon), \\ a_3 &= 2\xi\eta \frac{\gamma-2}{(\gamma-1)c^2(\rho)}, & b_1 &= -2\xi, & b_2 &= -2\eta. \end{aligned}$$

4. Define $v = u - (\xi^2 + \eta^2) - \varphi$. Using $Lu = 0$, we have

$$\begin{aligned} 0 &= Lu \\ &= \sum_{i,j=1}^2 a_{ij} D_{ij}(v + \xi^2 + \eta^2 + \varphi) + \sum_{i=1}^2 a_i |D_i(v + \xi^2 + \eta^2 + \varphi)|^2 \\ &\quad + a_3 \sum_{i,j=1}^2 D_i(v + \xi^2 + \eta^2 + \varphi) D_j(v + \xi^2 + \eta^2 + \varphi) + \sum_{i=1}^2 b_i D_i(v + \xi^2 + \eta^2 + \varphi) \\ &= L_1 v + L_2 \varphi, \end{aligned}$$

where

$$\begin{aligned} L_1 v &= \sum_{i,j=1}^2 a_{ij} D_{ij}v + \sum_{i=1}^2 (a_i |D_i v|^2 + b_i D_i v) \\ &\quad + 4a_1 \xi D_\xi v + 2a_1 D_\xi v D_\xi \varphi + 4a_2 \xi D_\eta v + 2a_2 D_\eta v D_\eta \varphi \\ &\quad + \frac{2(\gamma-2)}{(\gamma-1)c^2(\rho)} \xi \eta (D_\xi v D_\eta v + 2\eta D_\xi v + D_\xi v D_\eta \varphi + 2\xi D_\eta v + D_\eta v D_\xi \varphi), \end{aligned}$$

and

$$\begin{aligned}
L_2\varphi &= \sum_{i,j=1}^2 a_{ij}D_{ij}\varphi + \sum_{i=1}^2 (a_i|D_i\varphi|^2 + b_iD_i\varphi) + 4a_1\xi D_\xi\varphi + 4a_2\eta D_\eta\varphi \\
&\quad + \frac{4(\gamma-2)}{(\gamma-1)c^2(\rho)}\xi\eta(\xi D_\xi\varphi + \eta D_\eta\varphi) + \frac{2(\gamma-2)}{(\gamma-1)c^2(\rho)}\xi\eta D_\xi\varphi D_\eta\varphi \\
&\quad + 2(a_{11} + a_{22} + 2(\xi^2 a_{11} + \eta^2 a_{22})) + \frac{4(\gamma-2)}{(\gamma-1)c^2(\rho)}\xi^2\eta^2 - 2(\xi^2 + \eta^2)).
\end{aligned}$$

We now evaluate $L_2\varphi$:

$$\begin{aligned}
L_2\varphi &= \delta \sum_{i,j=1}^2 a_{ij}\tau(\zeta^{\tau-1}D_{ii}\zeta + \zeta^{\tau-2}(\tau-1)|D_i\zeta|^2) \\
&\quad + \delta \sum_{i=1}^2 (\delta a_i\tau^2\zeta^{2\tau-2}|D_i\zeta|^2 + b_i\tau\zeta^{\tau-1}D_i\zeta) + 4\delta a_1\xi\tau\zeta^{\tau-1}D_\xi\zeta \\
&\quad + 4\delta a_2\eta\tau\zeta^{\tau-1}D_\eta\zeta + \frac{4(\gamma-2)}{(\gamma-1)\rho^{\gamma-1}}\delta\xi\eta\tau\zeta^{\tau-1}(\xi D_\xi\zeta + \eta D_\eta\zeta) \\
&\quad + \frac{2(\gamma-2)}{(\gamma-1)\rho^{\gamma-1}}\delta^2\xi\eta\tau^2\zeta^{2\tau-2}D_\xi\zeta D_\eta\zeta \\
&\quad + 2\left(a_{11} + a_{22} + 2(\xi^2 a_{11} + \eta^2 a_{22}) + \frac{4(\gamma-2)}{(\gamma-1)c^2(\rho)}\xi^2\eta^2 - 4(\xi^2 + \eta^2)\right).
\end{aligned}$$

Noting that, for

$$\varepsilon \leq \varepsilon_0 = \frac{\bar{c}^2(\bar{\rho}, \rho_0)}{2\left|1 - \frac{2(\gamma-2)c^2(\rho_1)}{(\gamma-1)\rho_0}\right|},$$

we have

$$\begin{aligned}
&2(a_{11} + a_{22} + 2(\xi^2 a_{11} + \eta^2 a_{22})) + \frac{4(\gamma-2)}{(\gamma-1)\rho^{\gamma-1}}\xi^2\eta^2 - 2(\xi^2 + \eta^2) \\
&\geq 2\left(\varepsilon(1 - 2(\gamma-2)\frac{\xi^2 + \eta^2}{\rho}) + \xi^2 + \eta^2 + 2(c^2 - \xi^2 - \eta^2)\right) \\
&\geq c^2(\rho_0) > 0.
\end{aligned}$$

Also, since $\rho_0 < \bar{\rho} \leq \rho \leq \rho_1$, and $D_i\zeta$ and $D_{ii}\zeta$, $i = 1, 2$, are bounded in $\tilde{\Omega}$, we can find δ_2 small enough such that

$$L_2\varphi \geq \frac{1}{2}c^2(\rho_0) > 0 \quad \text{for } \delta \leq \delta_2.$$

5. In addition, if $B_R(X_0) \cap \Gamma_{\text{shock}} \neq \emptyset$, we have

$$0 = (\gamma-1)\rho^{\gamma-2}N\rho = Nu = N(v + \xi^2 + \eta^2 + \varphi)$$

and, in $O = \{c^2 - \xi^2 - \eta^2 < \varphi\} \subset B_R(X_0)$, we have

$$\begin{aligned} N(\xi^2 + \eta^2) &= 2\xi\beta_1 + 2\eta\beta_2 \\ &= (-\eta'\xi + \eta)(\xi + \eta\eta')(c^2(c^2 - \bar{c}^2) - \varphi(c^2 + 3\bar{c}^2)) \\ &\leq (-\eta'\xi + \eta)(\xi + \eta\eta')(c^2(c^2 - \bar{c}^2) - \delta\zeta^\tau(c^2 + 3\bar{c}^2)). \end{aligned}$$

Thus, in $O := \{c^2 - \xi^2 - \eta^2 < \varphi\} \subset B_R(X_0)$, by choice of $\delta \leq \delta_3(\delta_1)$ sufficiently small, we obtain

$$\begin{aligned} N(\xi^2 + \eta^2 + \varphi) &= 2\xi\beta_1 + 2\eta\beta_2 + \delta\tau(\beta_1) \\ &\leq (\eta - \eta'\xi)(\xi + \eta\eta')(c^2(c^2 - \bar{c}^2) - \delta\zeta^\tau(c^2 + 3\bar{c}^2)) + \delta\tau\zeta^{\tau-1}\|\beta\|_0 \\ &< 0. \end{aligned}$$

6. Finally, we cover $\Gamma_0 \cap \tilde{\Omega}$ with B_R centered at Γ_0 so that $D\varphi \cdot \nu = 0$. Hence, in $O \subset B_R(X_0)$ with $\delta_0 \leq \min\{\delta_1, \delta_2, \delta_3\}$ which depends on $\bar{\rho}$ and ρ_2 , but is independent of ε and $\tau \leq 2$, we have

$$\begin{aligned} L_1 v &= -L_2 \varphi \leq 0, \\ Nv &= -N(\xi^2 + \eta^2 + \varphi) \geq 0, \\ Dv \cdot \nu &= 0. \end{aligned}$$

Hence, we apply the weak maximum principle on $O \subset B_R(X_0)$ to obtain

$$(4.3) \quad \inf_O (c^2(\rho) - \xi^2 - \eta^2 - \varphi) \geq \inf_{\partial O} (c^2(\rho) - \xi^2 - \eta^2 - \varphi)^- = 0.$$

7. By piecing together these $B_{\frac{3}{4}R_{X_0}}(X_0)$, $X_0 \in \overline{\Omega} \setminus \Gamma_{\text{sonic}}$, we obtain a local uniform lower barrier of $c^2(\rho^\varepsilon) - (\xi^2 + \eta^2)$. That is,

$$c^2(\rho) - \xi^2 - \eta^2 \geq \varphi = \delta_0 \zeta^\tau \quad \text{in } B_{\frac{3}{4}R_{X_0}}(X_0) \cap \overline{\Omega_\varepsilon},$$

where δ_0 and τ are independent of ε (though they may depend on R). Moreover, δ_0 tends to 0 as $\text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \rightarrow 0$, so does φ . This completes the proof. \square

The proof of Lemma 4.1 also implies that we can obtain the uniform ellipticity of (3.2) which is independent of ε in $B_{\frac{3}{4}R_{X_0}}(X_0) \cap \overline{\Omega_\varepsilon}$.

The uniform lower bound of $c^2 - \xi^2 - \eta^2$ independent of ε implies that the governing equation (3.2) is locally uniformly elliptic, independent of ε and δ , which allows us to apply the standard local compactness arguments to obtain the limit ρ locally in the interior of the domain.

We first consider the behavior of shock position $r^{\varepsilon, \delta}$, as ε and δ tend to 0. We divide the shock position into three cases:

$$\text{Case 1: } c(\rho_0) < r(\theta) \leq \bar{c}(\rho_1, \rho_0) \text{ for all } \theta \in [\theta_w, \theta_1) \text{ and } r'(\theta) = r\sqrt{\frac{r^2 - \bar{c}^2}{\bar{c}^2}};$$

$$\text{Case 2: } r(\theta_w) = c(\rho_0) \text{ and } c(\rho_0) < r(\theta) \leq \bar{c}(\rho_1, \rho_0), \quad r'(\theta) = r\sqrt{\frac{r^2 - \bar{c}^2}{\bar{c}^2}} \text{ for all } \theta \in (\theta_w, \theta_1);$$

Case 3: There exists $\theta_a \in (\theta_w, \theta_1)$ such that $r(\theta) \equiv c(\rho_0)$ for $\theta \in [\theta_w, \theta_a]$, $r(\theta) > c(\rho_0)$, and $r'(\theta) = r\sqrt{\frac{r^2 - \bar{c}^2}{\bar{c}^2}}$ for $\theta \in (\theta_a, \theta_1)$.

Lemma 4.2. *There exist functions $r(\theta) \in C^1([\theta_w, \theta_1])$ and $\rho \in C_{loc}^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$, satisfying one of the three cases stated above, such that*

$$r^{\varepsilon, \delta} \rightarrow r \text{ in } C([\theta_w, \theta_1]), \quad \rho^{\varepsilon, \delta} \rightarrow \rho \text{ in } C_{loc}^{2+\alpha},$$

and (ρ, r) is a solution of the free boundary problem (3.2)–(3.7).

Proof. For $\varepsilon, \delta > 0$, it follows from Lemma 3.10 that

$$r^{\varepsilon, \delta} \in C^{1+\alpha}([\theta_w, \theta_1]), \quad \|r^{\varepsilon, \delta}\|_{C^1([\theta_w, \theta_1])} \leq C,$$

where C is independent of ε and δ . Thus, by the Ascoli-Arzelà theorem, there exists a subsequence converging uniformly to a function $r(\theta)$ in $C^\alpha([\theta_w, \theta_1])$ as $\varepsilon, \delta \rightarrow 0$ for any $\alpha < 1$. By the local ellipticity (cf. Lemma 4.1) and the standard interior Schauder estimate, there exists a function $\rho \in C_{loc}^{2+\alpha}$ such that $\rho^{\varepsilon, \delta} \rightarrow \rho$ in any compact subset contained by $\bar{\Omega} \setminus (\Gamma_{\text{sonic}} \cup \Gamma_{\text{shock}})$, satisfying $Q\rho = 0$ in Ω .

For $(r(\theta_0), \theta_0) \in \Gamma_{\text{sock}}$ with $r(\theta_0) > c(\rho_0)$, there exists a neighborhood of θ_0 and $\delta^* > 0$ independent of ε and δ such that $r^{\varepsilon, \delta} \geq c(\rho_0) + \delta^*$ for ε and δ small enough. It follows from $c(\rho^{\varepsilon, \delta}) \geq r^{\varepsilon, \delta} \geq c(\rho_0) + \delta^*$ that

$$\rho^{\varepsilon, \delta} > \rho_0 + \delta^*.$$

Thus, we obtain the uniform ellipticity locally, as well as uniform negativity of $\beta \cdot \nu$ locally. Hence, we can pass the limit to obtain $\rho \in C^{1+\alpha}$ and

$$M\rho = 0 \quad \text{on } \Gamma_{\text{shock}} \text{ near } (r(\theta_0), \theta_0)$$

such that $r'(\theta) = \frac{r}{\bar{c}}\sqrt{r^2 - \bar{c}^2}$.

Then the remainder is to show the case that $(r(\theta_0), \theta_0) \in \Gamma_{\text{shock}}$ and $r(\theta_0) = c(\rho_0)$.

First, it follows from Lemma 3.9 that

$$c(\rho_0) \leq r^{\varepsilon, \delta}(\theta) \leq c(\rho^{\varepsilon, \delta}(r^{\varepsilon, \delta}(\theta), \theta)) \leq c(\rho^{\varepsilon, \delta}(r^{\varepsilon, \delta}(\theta_0), \theta_0))$$

for $\theta \in [\theta_w, \theta_0]$, and

$$c(\rho_0) \leq \bar{c}(\rho^{\varepsilon, \delta}(r^{\varepsilon, \delta}(\theta_0), \theta_0), \rho_0) \leq r^{\varepsilon, \delta}(\theta_0).$$

Thus,

$$\rho^{\varepsilon, \delta}(r^{\varepsilon, \delta}(\theta_0), \theta_0) \rightarrow \rho_0.$$

Therefore, $r(\theta) \equiv c(\rho_0)$ for $\theta \in [\theta_w, \theta_0]$.

Next we prove the continuity of solutions up to the boundary where $r(\theta) = c(\rho_0)$. First, we prove that $r \in C^1$. Still from Lemma 3.9, we easily obtain that

$$\rho(r(\theta), \theta) \rightarrow \rho_0 \quad \text{if } \theta \rightarrow \theta_0 \text{ from the right.}$$

On the other hand,

$$r'(\theta) = r(\theta)\sqrt{\frac{r^2(\theta) - \bar{c}^2}{\bar{c}^2}} \quad \text{for } \theta > \theta_0,$$

which implies that $r'(\theta) \rightarrow 0$ as $\theta \rightarrow \theta_0$ from the right hand side, and it holds from the left hand side obviously. If we define $r'(\theta_0) = 0$, then $r \in C^1$.

Note that the equation for $u = c^2(\rho)$ is

$$(4.4) \quad \begin{aligned} Q(u) &:= (c^2 - r^2 + \varepsilon)u_{rr} + \frac{c^2 + \varepsilon}{r^2}u_{\theta\theta} + \frac{c^2 + (\gamma - 2)(r^2 - \varepsilon)}{(\gamma - 1)c^2}(u_r)^2 \\ &\quad + \frac{1}{(\gamma - 1)r^2}(u_\theta)^2 + \frac{c^2 - 2r^2 + \varepsilon}{r}u_r \\ &= 0. \end{aligned}$$

We prove the most complicated case $\theta_0 = \theta_a$ first, and the other cases will be discussed later.

We construct a family of barrier functions $\{\Psi_\tau\}$ with parameter τ . For any $m > 0$, there exists $\delta_1(m) > 0$ such that $r'(\theta) < m$ for $|\theta - \theta_a| < \delta_1(m)$. This implies that

$$|r(\theta) - r(\theta_a)| < m\delta_1(m) \quad \text{for } |\theta - \theta_a| < \delta_1(m),$$

where $\delta_1(m) \rightarrow 0$ as $m \rightarrow 0$.

Let $m < 1$ and $m\delta(m) = \frac{\tau}{2}$ (τ will be specified later). We have

$$\rho_0 \leq \rho(r(\theta), \theta) \leq \rho(r(\theta_a + \delta_1(m)), \theta_a + \delta_1(m)) \leq (\bar{c}_{\rho_0})^{-1}(r(\theta_a + \delta_1(m))) \leq \rho_0 + \frac{Cm}{2}.$$

For ε, δ small enough, we obtain

$$\rho_0 \leq \rho^{\varepsilon, \delta}(r^{\varepsilon, \delta}(\theta), \theta) \leq \rho_0 + Cm$$

and

$$0 \leq r^{\varepsilon, \delta}(\theta) - c(\rho_0) < \tau \quad \text{for } |\theta - \theta_a| < \delta_1(m),$$

where C depends only on γ and ρ_0 . We define

$$\Psi_\tau = \Psi_\tau^{\varepsilon, \delta} = c^2(\rho_0 + Cm) + A(c(\rho_0) + \tau - r)^\alpha + B(\theta - \theta_a)^2$$

in

$$Q^{\varepsilon, \delta} = \{(r, \theta) : |r - c(\rho_0)| \leq \delta_2, |\theta - \theta_a| \leq \delta_1(m)\} \cap \Omega^{\varepsilon, \delta},$$

where $\delta_2 > \tau$ will be chosen later.

Choose

$$B = \frac{c^2(\rho_1) - c^2(\rho_0)}{\delta_1^2(m)}, \quad A = A_1 = \frac{c^2(\rho_1) - c^2(\rho_0)}{\delta_2^\alpha}.$$

Since $\rho_0 \leq \rho^{\varepsilon, \delta} \leq \rho_1$ and $\rho^{\varepsilon, \delta} \leq \rho_0 + Cm$ on $\Gamma_{\text{shock}}^{\varepsilon, \delta} \cap \partial Q^{\varepsilon, \delta}$, we have

$$\Psi_\tau \geq c^2(\rho^{\varepsilon, \delta}) \quad \text{on } \partial Q^{\varepsilon, \delta}.$$

Thus, we have

$$(4.5) \quad \begin{aligned} Q(\Psi_\tau^{\varepsilon, \delta}) &= A\alpha(\alpha - 1)(c^2(\rho^{\varepsilon, \delta}) - r^2)(c(\rho_0) - r + \tau)^{\alpha-2} + \frac{2Bc^2}{r^2} \\ &\quad + (1 - \frac{\gamma-2}{(\gamma-1)c^2}(c^2 - r^2))A^2\alpha^2(c(\rho_0) - r + \tau)^{2\alpha-2} \\ &\quad + \frac{4B^2}{(\gamma-1)r^2}(\theta - \theta_a)^2 - \frac{A\alpha(c^2 - 2r^2)}{r}(c(\rho_0) - r + \tau)^{\alpha-1}. \end{aligned}$$

Consider (4.5) in $Q^{\varepsilon,\delta} \cap \{(r, \theta) : c^2(\rho^{\varepsilon,\delta}) - \Psi_\tau^{\varepsilon,\delta} \geq 0\}$. Since

$$c(\rho_0 + Cm) \geq c(\rho^{\varepsilon,\delta}(r^{\varepsilon,\delta}(\theta), \theta)) \geq r^{\varepsilon,\delta}(\theta) \geq r,$$

we have

$$\begin{aligned} c^2(\rho^{\varepsilon,\delta}) - r^2 + \varepsilon &\geq c^2(\rho^{\varepsilon,\delta}) - \Psi_\tau^{\varepsilon,\delta} + c^2(\rho_0 + Cm) - r^2 + A(c(\rho_0) + \tau - r)^\alpha \\ &\geq A(c(\rho_0) + \tau - r)^\alpha. \end{aligned}$$

For $\alpha < 1$, (4.5) implies

$$\begin{aligned} Q(\Psi_\tau^{\varepsilon,\delta}) &\leq A^2 \alpha \left(\left(2 - \frac{\gamma-2}{(\gamma-1)c^2} (c^2 - r^2) \right) \alpha - 1 \right) (c(\rho_1) - r + \tau)^{2\alpha-2} \\ &\quad - \frac{c^2 - 2r^2}{r} A \alpha (c(\rho_1) - r + \tau)^{\alpha-1} + \frac{4}{(\gamma-1)r^2} B^2 (\theta - \theta_a)^2 + 2B \frac{c^2}{r^2}. \end{aligned}$$

Moreover, let

$$A > A_1, \quad B = \frac{C(\rho_0, \rho_1)}{\delta_1^2(m)}.$$

If $\alpha < \frac{1}{2+C(\rho_0, \rho_1, \gamma)}$ and $\delta_2 + \tau$ small enough, we have

$$Q(\Psi_\tau^{\varepsilon,\delta}) \leq C(\rho_0, \rho_1) \left((2 + C(\rho_0, \rho_1, \gamma)) \alpha - 1 \right) A^2 (c(\rho_1) + \tau - r)^{2\alpha-2} + \frac{C(\rho_1, \rho_2)}{\delta_1^2(m)}.$$

Then there exists a constant $A_2(\delta_2, m, \rho_0, \rho_1)$ such that

$$Q(\Psi_\tau^{\varepsilon,\delta}) \leq 0 \quad \text{for } A > A_2.$$

In fact, if $r < c(\rho_0)$, we choose $\delta_2 = \sqrt{m}\delta_1(m)$ to obtain

$$c(\rho_0) - r + \tau \leq 2\sqrt{m}\delta_1(m),$$

and let

$$A_2^{(1)} = \frac{C(\rho_0, \rho_1, \alpha)m^{\frac{1-\alpha}{2}}}{\delta_1^\alpha(m)}, \quad A_1 = \frac{C(\rho_0, \rho_1)}{m^{\frac{\alpha}{2}}\delta_1^\alpha(m)}.$$

If $r \geq c(\rho_0)$,

$$c(\rho_0) + \tau - r \leq \tau,$$

and we let

$$A_2^{(2)} = \frac{C(\rho_0, \rho_1, \alpha)m^{1-\alpha}}{\delta_1^\alpha(m)}.$$

Set $A = \max\{A_1, A_2^{(1)}, A_2^{(2)}\}$. Then

$$\rho^{\varepsilon,\delta} \leq \Psi_\tau^{\varepsilon,\delta} \quad \text{in } Q^{\varepsilon,\delta}.$$

Passing to the limits $\delta, \varepsilon \rightarrow 0$, we obtain

$$\rho \leq \Psi_\tau \quad \text{in the domain } Q(m, \sqrt{m}\delta_1(m)) := \cap_{\delta, \varepsilon > 0} Q^{\varepsilon,\delta}.$$

With these barrier functions, we can show that ρ is continuous at $(r(\theta_a), \theta_a)$. In fact, for every $\varepsilon_1 > 0$, there exists $m > 0$ such that

$$c^2(\rho_0 + Cm) - c^2(\rho_0) < \frac{\varepsilon_1}{3}.$$

For this m , we can choose A , B , and τ such that

$$c^2(\rho) \leq \Psi_\tau \leq \frac{\varepsilon_1}{3} + c^2(\rho_1) + A(c(\rho_0) - r + \tau)^\alpha + B(\theta - \theta_a)^2.$$

Choose the neighborhood of $(r(\theta_a), \theta_a)$ small enough so that

$$A(c(\rho_0) - r + \tau)^\alpha \leq A(2\tau)^\alpha \leq C(\rho_0, \rho_1, \alpha)m^{\frac{\alpha}{2}}.$$

Then, choosing m small again, we have

$$c^2(\rho) \leq \frac{2\varepsilon_1}{3} + c^2(\rho_0) + B(\theta - \theta_a)^2.$$

Finally, we choose a small neighborhood such that

$$c^2(\rho_0) \leq c^2(\rho) \leq \varepsilon_1 + c^2(\rho_0).$$

Thus, we obtain our claim that ρ is continuous at $(r(\theta_a), \theta_a)$, that is, the results hold for this case.

As for the case $\theta \in [\theta_w, \theta_a)$, we can choose arbitrary $\tau > 0$, which is independent of the neighborhood of θ . This fact makes the similar proof of this case much easier for all sufficiently small ε and δ , and we omit the details here. \square

Next, we discuss the wave strength at the sonic circle $r \equiv c(\rho_0)$ and conclude that Case 3 in Lemma 4.2 can not actually occur.

Lemma 4.3. *Let $r(\theta)$ be monotone increasing in θ on Γ_{shock} and $\rho > \rho_0$ in the subsonic region. Then $r(\theta) > c(\rho_0)$ for $\theta_w < \theta \leq \theta_1$.*

Proof. We divide the proof into five steps.

1. We show our claim by contradiction. More precisely, if there exists $\bar{\theta}$ such that $r(\bar{\theta}) = c(\rho_0) := c_0$. Then, using the monotonicity of $r(\theta)$,

$$r(\theta) \equiv c_0 \quad \text{for } \theta_w \leq \theta \leq \bar{\theta}.$$

2. For $\theta_0 \in [\theta_w, \bar{\theta}]$, we define

$$w_1 = c_0^2 + A_1(c_0 - r)^{\frac{1}{2}} - B_1(c_0 - r)^{\beta_1} + D_1(\theta - \theta_0)^2,$$

where $A_1, B_1, D_1 > 0$ and $\frac{1}{2} < \beta_1 < 1$, all of which will be specified later to proof that $\rho \in C^{\frac{1}{2}}$ near this boundary point.

Using (4.4) with the coefficient of u_{rr} replaced by $u - r^2$, we have

$$\begin{aligned} \hat{Q}(w_1) &= \left(-(\beta_1^2 - \frac{1}{4})A_1B_1(c_0 - r)^{\beta_1 - \frac{3}{2}} + O_1 \right) \\ &\quad + \left(\beta_1(2\beta_1 - 1)B_1^2(c_0 - r)^{2\beta_1 - 2} + O_2 \right) \\ &\quad - \frac{(\gamma-2)}{4(\gamma-1)c^2}A_1^2(c^2 - r^2)(c_0 - r)^{-1} \\ &\quad + \left(-\frac{A_1D_1}{4}(c_0 - r)^{-\frac{3}{2}}(\theta - \theta_0)^2 + O_4 \right), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
O_1 &= -\frac{A_1}{2}c_0(c_0 - r)^{-\frac{1}{2}} - 2c_0\beta_1(\beta_1 - 1)B_1(c_0 - r)^{\beta_1-1} + \frac{A_1}{4}(c_0 - r)^{\frac{1}{2}} \\
&\quad + \beta_1(\beta_1 - 1)B_1(c_0 - r)^{\beta_1} + \frac{2c^2}{r^2}D_1 + \frac{A_1r}{2}(c_0 - r)^{-\frac{1}{2}} - \beta_1B_1r(c_0 - r)^{\beta_1-1} \\
&\quad + \frac{(\gamma-2)\beta_1A_1B_1}{(\gamma-1)c^2}(c^2 - r^2)(c_0 - r)^{\beta_1-\frac{3}{2}} - \frac{A_1}{2r}(c^2 - r^2)(c_0 - r)^{\frac{1}{2}} \\
&\quad + \frac{\beta_1B_1}{r}(c^2 - r^2)(c_0 - r)^{\beta_1-1}, \\
O_2 &= -\frac{(\gamma-2)\beta_1^2B_1^2}{(\gamma-1)c^2}(c^2 - r^2)(c_0 - r)^{2\beta_1-2}, \\
O_4 &= \beta_1(\beta_1 - 1)B_1D_1(c_0 - r)^{\beta_1-2}(\theta - \theta_0)^2 + \frac{4D_1}{(\gamma-1)r^2}(\theta - \theta_0)^2.
\end{aligned}$$

Notice that there exists $0 < \alpha < \frac{1}{2}$ such that $c^2 - r^2 \leq (c_0 - r)^\alpha$ for $c_0 - r > 0$ small. Thus,

$$\left| \frac{(\gamma-2)}{4(\gamma-1)c^2}A_1^2(c^2 - r^2)(c_0 - r)^{-1} \right| \leq C(\rho_0, \rho_1)A_1^2(c_0 - r)^{\alpha-1}.$$

We can choose a proper constant α such that

$$\beta_1 - \frac{3}{2} < \alpha - 1, \quad \text{i.e.,} \quad \alpha > \beta_1 - \frac{1}{2}.$$

On one hand, let $c_0 - r > 0$ be small enough so that

$$(\beta_1^2 - \frac{1}{4})A_1B_1(c_0 - r)^{\beta_1-\frac{3}{2}} > 3C(\rho_0, \rho_1)A_1^2(c_0 - r)^{\alpha-1},$$

which implies

$$(4.7) \quad B_1 > \frac{3C(\rho_0, \rho_1)}{\beta_1^2 - \frac{1}{4}}A_1(c_0 - r)^{\alpha-\beta_1+\frac{1}{2}} := A_1C(\rho_0, \rho_1, \beta_1)(c_0 - r)^{\alpha-\beta_1+\frac{1}{2}}.$$

On the other hand, if $c_0 - r > 0$ is sufficiently small, we have

$$(\beta_1^2 - \frac{1}{4})A_1B_1(c_0 - r)^{\beta_1-\frac{3}{2}} > 3\beta_1(2\beta_1 - 1)B_1^2(c_0 - r)^{2\beta_1-2},$$

which implies

$$(4.8) \quad A > \frac{(2\beta_1^2 - \beta_1)B_1}{\beta_1^2 - \frac{1}{4}}(c_0 - r)^{\beta_1-\frac{1}{2}} := C(\beta_1)B_1(c_0 - r)^{\beta_1-\frac{1}{2}}.$$

Moreover, we have

$$C(\rho_0, \rho_1, \beta_1)(c_0 - r)^{\alpha-\beta_1+\frac{1}{2}} < C(\beta_1)(c_0 - r)^{\frac{1}{2}-\beta_1}$$

when $r \in [\bar{r}, c_0]$ and \bar{r} is close to c_0 .

Choose proper constants A_1 and D_1 such that

$$w_1 > c_0^2 + \frac{1}{2}A_1(c_0 - r)^{\frac{1}{2}} + D_1(\theta - \theta_0)^2 > c^2$$

at the boundary of a relatively neighborhood N_1 of (c_0, θ_0) to Ω . Choose B_1 sufficiently small such that

$$\hat{Q}(w_1) < 0$$

and

$$C(\rho_0, \rho_1, \beta_1)(c_0 - r)^{\alpha - \beta + \frac{1}{2}} < \frac{B_1}{A_1} < \min \left\{ C(\beta), \frac{1}{4(\beta_1 - \beta_1^2)} \right\} (c_0 - r)^{\frac{1}{2} - \beta_1} \quad \text{in } N_1.$$

This implies that (4.7) and (4.8) hold.

Obviously, we have

$$\partial_{rr} w_1 < 0 \quad \text{in } N_1$$

if (4.7) and (4.8) hold. If $S_1 = \{(r, \theta) \in N_1 : c^2 > w_1\} \neq \emptyset$, we have $Q(w_1) \leq \hat{Q}(w_1) < 0$ in S_1 . Thus,

$$0 < Qu - Q(w_1).$$

Using the maximum principle,

$$u \leq w_1,$$

which contradicts with $c^2 > w_1$. Thus

$$c^2 \leq w_1 \quad \text{in } N_1.$$

3. We define

$$w_2 = c_0^2 + A_2(c_0 - r)^{\frac{1}{2}} + B_2(c_0 - r)^{\beta_2} - D_2(\theta - \theta_0)^2,$$

where $A_2, B_2, D_2 > 0$ and $\frac{1}{2} < \beta_2 < 1$, all of which will be specified later to prove that $C^{\frac{1}{2}}$ is optimal. Through a simple algebraic calculation, we have

$$\begin{aligned} \hat{Q}(w_2) &= ((\beta_2^2 - \frac{1}{4})A_2B_2(c_0 - r)^{\beta_2 - \frac{3}{2}} + \overline{O}_1) + (\beta_2(2\beta_2 - 1)B_2^2(c_0 - r)^{2\beta_2 - 2} + \overline{O}_2) \\ &\quad + (-\beta_2(\beta_2 - 1)B_2D_2(c_0 - r)^{\beta_2 - 2}(\theta - \theta_0)^2 + \overline{O}_3) \\ &\quad + (\beta_2B_2r(c_0 - r)^{\beta_2 - 1} + \overline{O}_4) + \frac{1}{4}A_2D_2(c_0 - r)^{-\frac{3}{2}}(\theta - \theta_0)^2 \\ &\quad + 2\beta_2(\beta_2 - 1)B_2c_0(c_0 - r)^{\beta_2 - 1}, \end{aligned}$$

where

$$\begin{aligned} \overline{O}_1 &= -\frac{(\gamma - 2)}{4(\gamma - 1)c^2}A_2^2(c^2 - r^2)(c_0 - r)^{-1} - \frac{A_2}{2}c_0(c_0 - r)^{-\frac{1}{2}} + \frac{A_2}{4}(c_0 - r)^{\frac{1}{2}} \\ &\quad - \beta_2(\beta_2 - 1)B_2(c_0 - r)^{\beta_2} - \frac{2c^2}{r^2}D_2 + \frac{A_2r}{2}(c_0 - r)^{-\frac{1}{2}} - \frac{A_2}{2r}(c^2 - r^2)(c_0 - r)^{\frac{1}{2}} \\ &\quad - \frac{(\gamma - 2)\beta_2A_2B_2}{(\gamma - 1)c^2}(c^2 - r^2)(c_0 - r)^{\beta_2 - \frac{3}{2}}, \\ \overline{O}_2 &= -\frac{(\gamma - 2)\beta_2^2B_2^2}{(\gamma - 1)c^2}(c^2 - r^2)(c_0 - r)^{2\beta_2 - 2}, \\ \overline{O}_3 &= -\frac{4D_2}{(\gamma - 1)r^2}(\theta - \theta_0)^2, \\ \overline{O}_4 &= -\frac{\beta_2B_2}{r}(c^2 - r^2)(c_0 - r)^{\beta_2 - 1}. \end{aligned}$$

Let D_2 be large enough such that $c^2 > w_2$ for some $\theta = \theta_a, \theta_b$. We choose $\tilde{r} < c_0$ such that

$$\begin{aligned} c^2 &> c_0^2 + 2A_2(c_0 - \tilde{r})^{\frac{1}{2}} - D_2(\theta - \theta_0)^2 \\ &\geq c_0 + A_2(c_0 - \tilde{r})^{\frac{1}{2}} + B_2(c_0 - \tilde{r})^{\beta_2} - D_2(\theta - \theta_0)^2. \end{aligned}$$

The second inequality holds provided that $\frac{B_2}{A_2} \leq (c_0 - \tilde{r})^{\frac{1}{2} - \beta_2}$. Choosing $\beta_2 > \frac{7}{8}$, we have

$$\frac{1}{2}\beta_2 B_2 r (c_0 - r)^{\beta_2 - 1} + 2\beta_2(\beta_2 - 1)B_2 c_0 (c_0 - r)^{\beta_2 - 1} \leq 0 \quad \text{for } \frac{c_0}{2} < r < c_0,$$

and

$$\hat{Q}(w_2) > 0.$$

Then, if $S_2 = \{(r, \theta) \in N_1 : c^2 < w_2\} \neq \emptyset$, we have

$$Q(w_2) \geq \hat{Q}(w_2) > 0 \quad \text{in } S_2.$$

Thus,

$$Qu - Q(w_2) < 0.$$

Using the maximum principle, $c \leq w_2$, which contradicts with $c^2 < w_1$. Thus

$$c^2 \geq w_2 \quad \text{in } N_2.$$

4. We now show that

$$c^2 > c_0^2 + A_3(c_0 - r)^{\frac{1}{2}} + B_3(c_0 - r)^{\beta_3} =: w_3$$

in a relative neighborhood of (r_0, θ_0) , where A_3 and B_3 are positive constants to be specified later, so that the $C^{\frac{1}{2}}$ -regularity is optimal.

Since $c^2 \geq w_2$, we can choose $\bar{\theta}_a$ and $\bar{\theta}_b$ such that

$$c^2 \geq c_0^2 + A_2(\bar{\theta}_a, \bar{\theta}_a)(c_0 - r)^{\frac{1}{2}} + B_2(c_0 - r)^{\beta_2} \quad \text{for } N_3 \subset N_2.$$

Thus, there exist positive constants A_3 , B_3 , and β_3 such that

$$w_3 \leq c^2.$$

It is easy to see that

$$\begin{aligned} \hat{Q}(w_3) &= ((\beta_3^2 - \frac{1}{4})A_3 B_3 (c_0 - r)^{\beta_3 - \frac{3}{2}} + \tilde{O}_1) + (\beta_3(2\beta_3 - 1)B_3^2 (c_0 - r)^{2\beta_3 - 2} + \tilde{O}_2) \\ &\quad + (\beta_3 B_3 r (c_0 - r)^{\beta_3 - 1} + \tilde{O}_4) + 2\beta_3(\beta_3 - 1)B_3 c_0 (c_0 - r)^{\beta_3 - 1}, \end{aligned}$$

where

$$\begin{aligned}
\tilde{O}_1 &= -\frac{(\gamma-2)}{4(\gamma-1)c^2}A_3^2(c^2-r^2)(c_0-r)^{-1} - \frac{A_3}{2}c_0(c_0-r)^{-\frac{1}{2}} + \frac{A_3}{4}(c_0-r)^{\frac{1}{2}} \\
&\quad + \frac{A_3r}{2}(c_0-r)^{-\frac{1}{2}} - \frac{A_3}{2r}(c^2-r^2)(c_0-r)^{\frac{1}{2}} \\
&\quad - \frac{(\gamma-2)\beta_3A_3B_3}{(\gamma-1)c^2}(c^2-r^2)(c_0-r)^{\beta_3-\frac{3}{2}}, \\
\tilde{O}_2 &= \beta_3(\beta_3-1)B_3(c_0-r)^{\beta_3} - \frac{(\gamma-2)\beta_3^2B_3^2}{(\gamma-1)c^2}(c^2-r^2)(c_0-r)^{2\beta_3-2}, \\
\tilde{O}_3 &= -\frac{\beta_3B_3}{r}(c^2-r^2)(c_0-r)^{\beta_3-1}.
\end{aligned}$$

Similarly, we can show that $c^2 \geq w_3$ in N_3 .

Thus, we have

$$\frac{1}{2}A_3(c_0-r)^{\frac{1}{2}} \leq c^2 - c_0^2 \leq 2A_1(c_0-r)^{\frac{1}{2}} \quad \text{in } N_1 \cap N_3.$$

This implies

$$a(c_0-r)^{\frac{1}{2}} \leq v := \rho - \rho_0 \leq A(c_0-r)^{\frac{1}{2}} \quad \text{in } N_1 \cap N_3$$

for some constants a and A , so the optimal regularity of ρ is $C^{\frac{1}{2}}$ near the sonic circle.

5. We introduce the coordinates:

$$(x, y) = (c_0 - r, \theta - \theta_w)$$

and set

$$v = c^2 - c_0^2.$$

Thus, rewriting the equation for c^2 in the divergence form, we have

$$(4.10) \quad Qv = (a_{11}(v + 2c_0x - x^2)v_x)_x + b_1v_x + (a_{22}v_y)_y = 0,$$

where $a_{11} = \frac{c^{2(2-\gamma)}}{\gamma-1}$, $a_{22} = \frac{c^{\frac{2}{\gamma-1}}}{(\gamma-1)r^2}$, and $b_1 = \frac{c^{\frac{2}{\gamma-1}}}{(\gamma-1)r}$.

Scaling v in $N_1 \cap N_3$ by defining

$$(4.11) \quad u(S, T) = \frac{1}{S^{\frac{1}{5}}}v(S^{-\frac{12}{5}}, y_0 + S^{-\frac{14}{5}}T),$$

for $(S^{-\frac{12}{5}}, y_0 + S^{-\frac{14}{5}}T) \in N_1 \cap N_3$. Moreover, u satisfies the following governing equation:

$$(4.12) \quad Qu = (\tilde{a}_{11}u_S)_S + (\tilde{a}_{12}u_T)_S + (\tilde{a}_{21}u_S)_T + (\tilde{a}_{22}u_T)_T + (\tilde{b}_2u)_T + \tilde{c}_1u_S + \tilde{c}_2u_T + \tilde{d}_2u = 0,$$

where

$$\begin{aligned}
\tilde{a}_{11} &= a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{12} &= \frac{14T}{5S}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{21} &= \frac{14T}{5S}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{22} &= \frac{144}{25}a_{22} + \frac{189T^2}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{b}_2 &= \frac{14T}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) = \tilde{b}_{22}TS^{-2}, \\
\tilde{c}_1 &= \frac{S^{\frac{7}{5}}u(2-\gamma)S^{-\frac{11}{5}}}{5(\gamma-1)^2}c^{\frac{2(3-2\gamma)}{\gamma-1}}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) \\
&\quad - \frac{a_{11}(4c_0 - S^{-\frac{12}{5}})}{5S^{\frac{11}{5}}} - \frac{12b_1}{5S^{\frac{11}{5}}} = \tilde{c}_{11}S^{-\frac{11}{5}}, \\
\tilde{c}_2 &= \frac{168T}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) - \frac{189Ta_{11}}{25S^2} - \frac{168b_1T}{25S^{\frac{16}{5}}} = \tilde{c}_{22}S^{-2}T, \\
\tilde{d} &= \frac{S^{\frac{7}{5}}u(2-\gamma)S^{-\frac{16}{5}}}{25(\gamma-1)^2}c^{\frac{2(3-2\gamma)}{\gamma-1}}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) \\
&\quad - \frac{13a_{11}(2c_0 - S^{-\frac{12}{5}})}{25S^{\frac{16}{5}}} - \frac{12a_{11}S^{\frac{7}{5}}u(c_0 + \frac{6}{5}S^{-\frac{6}{5}})}{25S^{\frac{23}{5}}} - \frac{12b_1}{25S^{\frac{16}{5}}} = \tilde{d}_1S^{-\frac{16}{5}}.
\end{aligned}$$

From the optimal continuity,

$$0 < a \leq S^{\frac{7}{5}}u \leq A,$$

we have

$$0 < C^{-1} \leq \lambda_1, \lambda_2, \tilde{b}_{22}, \tilde{c}_{11}, \tilde{c}_{22}, \tilde{d}_1 \leq C$$

if S^{-1} and T are sufficiently small. Here λ_1 and λ_2 are the eigenvalues of the matrix $(\tilde{a}_{ij})_{2 \times 2}$, so the equation is uniformly elliptic for u in the (S, T) -coordinates.

Let $x_0^{-1} < S \leq x_0^{-\frac{5}{4}}$ with x_0 small enough. Then, using Theorem 8.20 in [16], we have

$$\begin{aligned}
ax_0^{\frac{7}{5}} \leq u(x_0^{-1}, 0) &\leq \sup_{x_0^{-1} \leq S \leq x_0^{-5/4}} u(S, T) \\
&\leq C \inf_{x_0^{-1} \leq S \leq x_0^{-5/4}} u(S, T) \leq Cu(x_0^{-\frac{5}{4}}, 0) \leq CAx_0^{\frac{7}{4}},
\end{aligned}$$

where $C \leq C(n)^{(\frac{\Lambda}{\lambda} + \nu R)}$ in [16] is independent of x_0 , since $(\Lambda, \lambda) = (\lambda_1, \lambda_2)$, $R = x_0^{-\frac{5}{4}} - x_0^{-1} \leq x_0^{-\frac{5}{4}}$, and $\nu := \max_{x_0^{-1} \leq S \leq x_0^{-5/4}} \{\tilde{b}_2, \tilde{c}_1, \tilde{c}_2, \sqrt{\tilde{d}}\} \leq Cx_0^{\frac{8}{5}}$. This implies that

$x_0^{-\frac{7}{20}} \leq C$, which is a contradiction if x_0 is sufficiently small. This completes the proof. \square

Next, we consider Γ_{shock} in the (ξ, η) -coordinates to obtain finer properties.

Lemma 4.4. *For the free boundary $\Gamma_{\text{shock}} = \{(\xi, \eta(\xi)) : \xi_w < \xi < \xi_1\}$ determined by (3.2)–(3.7), we have*

$$\eta(\xi) \in C^2([\xi_w, \xi_1])$$

and $\eta(\xi)$ is strictly convex for $\xi \in [\xi_w, \xi_1]$.

Proof. We define

$$(4.13) \quad F(\xi, \eta) = \xi^2 + \eta^2 - r^2(\theta(\xi, \eta)) = 0 \quad \text{on } \Gamma_{\text{shock}}.$$

It is easy to check that

$$F_\eta = (2\eta - 2rr'\theta_\eta)|_{\xi=\xi_w} = 2\eta(\xi_w) \neq 0.$$

By the implicit function theorem, there exists $\eta = \eta(\xi)$ such that (4.13) holds locally on Γ_{shock} near $\xi = \xi_w$. That is, there exists $\bar{\xi} > 0$ such that $(\xi, \eta(\xi)) \in \Gamma_{\text{shock}}$ for $\xi_w < \xi \leq \bar{\xi}$.

Recall that $\eta'(\xi) = f(\xi, \eta(\xi), \rho(\xi, \eta(\xi)))$. Then

$$\eta'' = f_\xi + f_\eta \eta' + f_\rho \rho' \quad \text{for } \xi \in (\xi_w, \bar{\xi}).$$

Observe that, if ρ were constant, the shock would be a straight line. We conclude

$$f_\xi + f_\eta \eta' = 0.$$

Therefore, the sign of η'' is determined entirely by the sign of f_ρ and ρ' . Note that ρ is increasing, $\rho' > 0$, and $\frac{d\bar{c}^2}{d\rho} > 0$. Moreover, we have

$$(4.14) \quad \begin{aligned} \frac{\partial f}{\partial \bar{c}^2} &= \frac{-2\xi\eta\bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2} + 2\eta^2(\xi^2 + \eta^2 - \bar{c}^2) + (\xi^2 + \eta^2)(\bar{c}^2 - \eta^2)}{\bar{c}(\xi\eta - \bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2\sqrt{\xi^2 + \eta^2 - \bar{c}^2}} \\ &= \frac{(\xi\bar{c} - \eta\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2}{\bar{c}(\xi\eta - \bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2\sqrt{\xi^2 + \eta^2 - \bar{c}^2}}. \end{aligned}$$

If $\xi\eta \leq 0$, it is clear from (4.14) that

$$\frac{\partial f}{\partial \bar{c}^2} > 0.$$

If $\xi\eta > 0$, from (4.14), we have

$$\begin{aligned} \frac{\partial f}{\partial \bar{c}^2} &= \frac{(\xi^2 + \eta^2)^2(\bar{c}^2 - \eta^2)^2(\xi\eta + \bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2}{\bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2}(\xi^2 - \bar{c}^2)(\eta^2 - \bar{c}^2)(\xi\bar{c} + \eta\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2} \\ &= \frac{(\xi^2 + \eta^2)^2(\xi\eta + \bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2}{\bar{c}(\bar{c}^2 - \xi^2)\sqrt{\xi^2 + \eta^2 - \bar{c}^2}(\xi\bar{c} + \eta\sqrt{\xi^2 + \eta^2 - \bar{c}^2})^2} > 0. \end{aligned}$$

These imply that $\eta = \eta(\xi)$ is strictly convex for $\xi \in [\xi_w, \xi_1]$. \square

Based on the lemma above, we have

Remark 4.5. Problem (3.2)–(3.7) is equivalent to the following free boundary problem in the self-similar coordinates:

(i) Equation:

$$(4.15) \quad L\rho = \sum_{i,j=1}^2 D_i(a_{ij}(\xi, \eta, \rho)D_j\rho) + \sum_{i=1}^2 b_i(\xi, \eta)D_i\rho = 0 \quad \text{in } \Omega$$

with

$$\begin{aligned} a_{11}(\xi, \eta, \rho) &= c^2(\rho) - \xi^2, & a_{22}(\xi, \eta, \rho) &= c^2(\rho) - \eta^2, \\ a_{12}(\xi, \eta, \rho) &= a_{21}(\xi, \eta, \rho) = -\xi\eta, & b_1(\xi, \eta) &= \xi, & b_2(\xi, \eta) &= \eta. \end{aligned}$$

(ii) The shock equation:

$$\frac{d\eta}{d\xi} = f(\xi, \eta, \rho) = \frac{\xi\eta + \bar{c}\sqrt{\xi^2 + \eta^2 - \bar{c}^2}}{\xi^2 - \bar{c}^2} \quad \text{with } \eta(\xi_1) = \eta_1,$$

with the boundary condition on Γ_{shock} :

$$(4.16) \quad N\rho = \sum_{i=1}^2 \beta_i D_i\rho = 0 \quad \text{on } \Gamma_{\text{shock}} = \{\eta = \eta(\xi) : 0 \leq \xi \leq \xi_1\},$$

where β_1 and β_2 are the following functions of (ξ, η) , ρ , and η' :

$$(4.17) \quad \begin{aligned} \beta_1 &= (\xi^2 + \eta^2)(-\eta'\xi + \eta)(c^2(\rho) + \bar{c}^2(\rho, \rho_0)) \\ &\quad - 2\bar{c}^2(\rho, \rho_0)(-\eta'\xi(c^2 + \eta^2) + (\eta - \eta(\eta')^2 - \xi\eta')(c^2 - \xi^2)) \end{aligned}$$

and

$$(4.18) \quad \begin{aligned} \beta_2 &= \eta'(\xi^2 + \eta^2)(\eta - \eta'\xi)(c^2(\rho) + \bar{c}^2(\rho, \rho_0)) \\ &\quad - 2\bar{c}^2(\rho, \rho_0)((\eta'\eta - \xi - \xi(\eta')^2)(c^2 - \eta^2) + \eta'\eta(c^2 + \xi^2)). \end{aligned}$$

(iii) The remaining boundary conditions:

$$(4.19) \quad \rho = \rho_2 \quad \text{on } \Gamma_{\text{sonic}}, \quad \rho_{\nu} = 0 \quad \text{on } \Gamma_0, \quad \rho(P_2) = \bar{\rho},$$

where ν is the outward normal to Ω at Γ_0 .

It is easy to check that (4.16) is the oblique derivative boundary condition along Γ_{shock} .

With Lemma 4.4, we can show that Case 1 is the only case for the solutions, which implies that we can obtain the finer regularity near P_2 .

Lemma 4.6. *Suppose that (ρ, r) is the solution to the free boundary problem (3.2)–(3.7). Then the shock does not meet the circle $r = r_0$ at the wedge.*

Proof. We show our claim by contradiction. Otherwise, $r(\theta_w) = c_0$.

1. First, let $\eta = r \cos(\theta - \theta_w)$ and consider

$$\phi = c_0^2 + A_1(c_0 - \eta)^{\frac{1}{2}} - B_1(c_0 - \eta)^{\beta_1} + C_1(\theta - \theta_0)^2,$$

where $\theta \in [\theta_w, \theta_w + \delta]$, $\delta > 0$ small enough, $A_1, B_1, C_1 > 0$ and $\frac{1}{2} < \beta_1 < 1$, all of which will be specified later to prove that $\rho \in C^{\frac{1}{2}}$ near this boundary point. We now show that ϕ is an upper barrier of ρ .

2. Since $0 \leq c_0^2 - \eta^2 = (c_0^2 - r^2) + r^2 \sin^2(\theta - \theta_w)$ on Γ_{shock} from its convexity indicated in Lemma 4.4, we have

$$0 \leq r^2 - c_0^2 \leq r^2 \sin^2(\theta - \theta_w) \leq C(\theta - \theta_w)^2 \quad \text{on } \Gamma_{\text{shock}}$$

for some positive constant C . This implies that

$$\bar{c}^2 - c_0^2 \leq r^2 - c^2 \leq r^2 - c_0^2 \leq C(\theta - \theta_w)^2.$$

Then

$$c^2 - c_0^2 \leq C(\theta - \theta_w)^2,$$

since c^2 and \bar{c}^2 are both functions of ρ . We can choose $C_1 > 0$ so large that $c^2 \leq \phi$ on Γ_{shock} . Moreover, we have

$$\begin{aligned} \phi_{\boldsymbol{\nu}}|_{\Gamma_{\text{wedge}}} &= \phi_{\theta}|_{\Gamma_{\text{wedge}}} \\ &= \left[\left(\frac{1}{2} A_1 (c_0 - \eta)^{-\frac{1}{2}} - \beta_1 B_1 (c_0 - \eta)^{\beta_1 - 1} \right) r \sin(\theta - \theta_w) + 2C_1(\theta - \theta_w) \right] \Big|_{\theta=\theta_w} \\ &= 0, \end{aligned}$$

where $\boldsymbol{\nu}$ denotes the unit normal. We can also choose proper A_1 and B_1 such that $c^2 \leq \phi$ in the remaining boundary of a neighborhood of (r_w, θ_w) as in Lemma 4.3.

3. Using (4.4) for Q with the coefficient of u_{rr} replaced by $u - r^2$, we obtain \hat{Q} . That is,

$$\begin{aligned} (4.20) \quad \hat{Q}(\phi) &:= \left(-\beta_1 B_1 \frac{c^2}{r} (c_0 - \eta)^{\beta_1 - 2} + O_1 \right) \\ &\quad - \frac{1}{4} A_1 (c_0 - \eta)^{-\frac{3}{2}} \left(C_1 (\theta - \theta_0)^2 \cos^2(\theta - \theta_w) + c^2 \sin^2(\theta - \theta_0) \right) - 4O_2, \end{aligned}$$

where

$$\begin{aligned}
O_1 = & -\frac{A_1}{4}(c^2 - r^2)(c_0 - \eta)^{-\frac{3}{2}} \cos^2(\theta - \theta_w) \\
& -\frac{(\gamma - 2)}{4(\gamma - 1)c^2} A_1^2 (c^2 - r^2)(c_0 - \eta)^{-1} \cos^2(\theta - \theta_w) \\
& -\beta_1(\beta_1 - 1)B_1(c^2 - r^2)(c_0 - \eta)^{\beta_1 - 2} \cos^2(\theta - \theta_w) \\
& -\frac{A_1^2}{4}(c_0 - \eta)^{-1} \cos^2(\theta - \theta_w) + \frac{A_1 c^2}{2r}(c_0 - \eta)^{-\frac{1}{2}} \cos^2(\theta - \theta_w) \\
& +\left(\frac{1}{4} - \beta_1(\beta_1 - 1)\right)A_1 B_1 (c_0 - \eta)^{\beta_1 - \frac{3}{2}} \cos^2(\theta - \theta_w) \\
& +\beta_1(2\beta_1 - 1)B_1^2 (c_0 - \eta)^{2\beta_1 - 2} \cos^2(\theta - \theta_w) - \beta_1 A_1 B_1 (c_0 - \eta)^{\beta_1 - \frac{3}{2}} \cos^2(\theta - \theta_w) \\
& -\frac{(\gamma - 2)A_1^2 B_1^2}{(\gamma - 1)c^2} (c^2 - r^2)(c_0 - \eta)^{\beta_1 - \frac{3}{2}} \cos^2(\theta - \theta_w) \\
& -\frac{A_1}{2r}(c^2 - 2r^2)(c_0 - \eta)^{-\frac{1}{2}} \cos(\theta - \theta_w) \\
& +\frac{\beta_1 B_1}{r}(c^2 - 2r^2)(c_0 - \eta)^{\beta_1 - 1} \cos(\theta - \theta_w) \\
& +\frac{(\gamma - 2)\beta_1^2 B_1^2}{(\gamma - 1)c^2} (c^2 - r^2)(c_0 - \eta)^{2\beta_1 - 2} \cos^2(\theta - \theta_w),
\end{aligned}$$

and

$$\begin{aligned}
O_2 = & \beta_1(\beta_1 - 1)B_1 C_1 (c_0 - \eta)^{\beta_1 - 2} (\theta - \theta_0)^2 \cos^2(\theta - \theta_w) \\
& -\beta_1(\beta_1 - 1)B_1 c^2 (c_0 - \eta)^{\beta_1 - 2} \sin^2(\theta - \theta_w) \\
& +\frac{1}{\gamma - 1} \beta_1^2 B_1^2 (c_0 - \eta)^{2\beta_1 - 2} \sin^2(\theta - \theta_w) + \frac{2c^2}{(\gamma - 1)r^2} (\theta - \theta_w)^2 \\
& -\frac{\beta_1 A_1 B_1}{\gamma - 1} (c_0 - \eta)^{\beta_1 - \frac{3}{2}} \sin^2(\theta - \theta_w) \\
& +\frac{A_1 C_1}{(\gamma - 1)r} (c_0 - \eta)^{-\frac{1}{2}} (\theta - \theta_w) \sin(\theta - \theta_w) \\
& -\frac{4\beta_1 B_1 C_1}{(\gamma - 1)r} (c_0 - \eta)^{\beta_1 - 1} (\theta - \theta_w) \sin(\theta - \theta_w).
\end{aligned}$$

It is easy to see that there exists a small neighborhood N_1 of (r_w, θ_w) such that $\hat{Q}(\phi) < 0$ in N_1 .

Obviously, we have $\partial_{rr}\phi < 0$ in N_1 .

4. Now we show that $c^2(\rho) \leq \phi$. If $S_1 = \{(r, \theta) \in N_1 : c^2 > \psi_1\} \neq \emptyset$, we have

$$Q(\psi_1) \leq \hat{Q}(\psi_1) < 0 \quad \text{in } S_1.$$

Thus, $Q(u) - Q(\psi_1) > 0$. Using the maximum principle, we have $u \leq \psi_1$. This contradicts with $c^2 > \psi_1$. Thus, $c^2 \leq \psi_1$ in N_1 , i.e.,

$$0 \leq c^2 - c_0^2 \leq A_1(c_0 - \eta)^{\frac{1}{2}} + C_1(\theta - \theta_w)^2.$$

5. Now we find a lower barrier for ρ .

As the proof of Lemma 4.3, we can show that there exist a neighborhood N_2 of (r_w, θ_w) and a positive constant A_2 such that

$$c^2 - c_0^2 \geq A_2(c_0 - r)^{\frac{1}{2}} \quad \text{in } N_2 \cap \{(r, \theta) : r \leq c_0\}.$$

The only new here is the boundary $r = c_0$, which is obvious. This implies that

$$a(c_0 - r)^{\frac{1}{2}} \leq v := \rho - \rho_0 \leq A(c_0 - r)^{\frac{1}{2}} \quad \text{in } N_1 \cap N_2 \cap V,$$

where V is an upward sector containing the wedge, with the vertex at P_2 and the angle smaller than $\frac{\pi}{2}$, for some constants a and A depending on V , so the optimal regularity along the wedge is $C^{\frac{1}{2}}$ near the sonic circle.

6. We introduce the coordinates:

$$(4.21) \quad x = c_0 - r, \quad y = \theta - \theta_w, \quad v = c^2 - c_0^2.$$

Thus, rewriting the equation for c^2 in the divergence form, we have

$$(4.22) \quad Qv = (a_{11}(v + 2c_0x - x^2)v_x)_x + b_1v_x + (a_{22}v_y)_y = 0,$$

where $a_{11} = \frac{c^{\frac{2(2-\gamma)}{\gamma-1}}}{\gamma-1}$, $a_{22} = \frac{c^{\frac{2}{\gamma-1}}}{\gamma-1} \frac{1}{r^2}$, and $b_1 = \frac{c^{\frac{2}{\gamma-1}}}{\gamma-1} \frac{1}{r}$.

Scale v in $N_1 \cap N_3 \cap V$ by defining

$$(4.23) \quad u(S, T) = \frac{1}{S^{\frac{1}{5}}} v(S^{-\frac{12}{5}}, S^{-\frac{14}{5}} T)$$

for $(S^{-\frac{12}{5}}, S^{-\frac{14}{5}} T) \in N_1 \cap N_3 \cap V$. Moreover, u satisfies the following governing equation:

$$(4.24) \quad Qu = 0,$$

where

$$(4.25) \quad Qu := (\tilde{a}_{11}u_S)_S + (\tilde{a}_{12}u_T)_S + (\tilde{a}_{21}u_S)_T + (\tilde{a}_{22}u_T)_T + (\tilde{b}_2u)_T + \tilde{c}_1u_S + \tilde{c}_2u_T + \tilde{d}_2u,$$

with

$$\begin{aligned}
\tilde{a}_{11} &= a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{12} &= \frac{14T}{5S}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{21} &= \frac{14T}{5S}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{a}_{22} &= \frac{144}{25}a_{22} + \frac{189T^2}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\
\tilde{b}_2 &= \frac{14T}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) = \tilde{b}_{22}TS^{-2}, \\
\tilde{c}_1 &= \frac{S^{\frac{7}{5}}u(2-\gamma)S^{-\frac{11}{5}}}{5(\gamma-1)^2}c^{\frac{2(3-2\gamma)}{(\gamma-1)}}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) \\
&\quad - \frac{a_{11}(4c_0 - S^{-\frac{12}{5}})}{5S^{\frac{11}{5}}} - \frac{12b_1}{5S^{\frac{11}{5}}} = \tilde{c}_{11}S^{-\frac{11}{5}}, \\
\tilde{c}_2 &= \frac{168T}{25S^2}a_{11}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) - \frac{189Ta_{11}}{25S^2} - \frac{168b_1T}{25S^{\frac{16}{5}}} = \tilde{c}_{22}S^{-2}T, \\
\tilde{d} &= \frac{S^{\frac{7}{5}}u(2-\gamma)S^{-\frac{16}{5}}}{25(\gamma-1)^2}c^{\frac{2(3-2\gamma)}{(\gamma-1)}}(S^{\frac{7}{5}}u + 2c_0S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) \\
&\quad - \frac{13a_{11}(2c_0 - S^{-\frac{12}{5}})}{25S^{\frac{16}{5}}} - \frac{12a_{11}S^{\frac{7}{5}}u(5c_0 + 6S^{-\frac{6}{5}})}{125S^{\frac{23}{5}}} - \frac{12b_1}{25S^{\frac{16}{5}}} = \tilde{d}_1S^{-\frac{16}{5}}.
\end{aligned}$$

From the optimal continuity, $0 < a \leq S^{\frac{7}{5}}u \leq A$. Then we have

$$0 < C^{-1} \leq \lambda_1, \lambda_2, \tilde{b}_{22}, \tilde{c}_{11}, \tilde{c}_{22}, \tilde{d}_1 \leq C$$

if S^{-1} and T are sufficiently small. Here λ_1 and λ_2 are the eigenvalues of the matrix $(\tilde{a}_{ij})_{2 \times 2}$, which implies that (4.24) is a uniform elliptic equation for u in the (S, T) -coordinates.

Let $x_0^{-1} < S \leq x_0^{-\frac{5}{4}}$ with x_0 small enough. Using Theorem 8.20 in [16], we have

$$\begin{aligned}
ax_0^{\frac{7}{5}} \leq u(x_0^{-1}, 0) &\leq \sup_{x_0^{-1} \leq S \leq x_0^{-5/4}} u(S, T) \\
&\leq C \inf_{x_0^{-1} \leq S \leq x_0^{-5/4}} u(S, T) \leq Cu(x_0^{-\frac{5}{4}}, 0) \leq CAx_0^{\frac{7}{4}},
\end{aligned}$$

where $C \leq C(n)^{\frac{\Lambda}{\lambda} + \mu R}$ in [16] is independent of x_0 here, since

$$\begin{aligned}
(\Lambda, \lambda) &= (\lambda_1, \lambda_2), \quad R = x_0^{-\frac{5}{4}} - x_0^{-1} \leq x_0^{-\frac{5}{4}}, \\
\mu &= \max_{x_0^{-1} \leq S \leq x_0^{-5/4}} \{\tilde{b}_2, \tilde{c}_1, \tilde{c}_2, \sqrt{\tilde{d}}\} \leq Cx_0^{\frac{8}{5}}.
\end{aligned}$$

This implies that

$$x_0^{-\frac{7}{20}} \leq C,$$

which is a contradiction if x_0 is small. This completes the proof. \square

Finally, we establish the Lipschitz continuity for the solution near the degenerate sonic boundary.

Lemma 4.7. *The solution ρ to the free boundary problem (4.15)–(4.19) is Lipschitz continuous up to the boundary Γ_{sonic} .*

Proof. On one hand, since $\rho \leq \rho_1$ in Ω , we have

$$c^2(\rho) - \xi^2 - \eta^2 < c^2(\rho_1) - \xi^2 - \eta^2.$$

On the other hand, it follows from Lemma 3.1 that

$$c^2(\rho) - \xi^2 - \eta^2 > \xi^2 + \eta^2 - c^2(\rho_1) \quad \text{in } \Omega.$$

Let $r_2^2 = c^2(\rho_2)$. We have

$$\begin{aligned} |c^2(\rho) - c^2(\rho_2)| &\leq |c^2(\rho) - \xi^2 - \eta^2| + |c^2(\rho_2) - \xi^2 - \eta^2| \\ &\leq 2|c^2(\rho_2) - \xi^2 - \eta^2| \\ &\leq 4r_2|r_2 - \sqrt{\xi^2 + \eta^2}|, \end{aligned}$$

which implies that ρ is Lipschitz continuous up to the degenerate boundary Γ_{sonic} . \square

Proof of the Existence Part of Theorem 2.1. The above seven lemmas, i.e., Lemma 4.1–Lemma 4.7, show that there exists a solution

$$(\rho, r) \in C^{2+\alpha}(\Omega) \cap C^\alpha(\overline{\Omega}) \cap C^{0,1}(\Omega \cup \Gamma_{\text{sonic}}) \times C^{2+\alpha'}((\theta_w, \theta_1)) \cap C^{1,1}([\theta_w, \theta_1])$$

which satisfies (2.11)–(2.15). This completes the proof of the existence part of Theorem 2.1. \square

5. PROOF OF THEOREM 2.1: OPTIMAL REGULARITY NEAR THE SONIC BOUNDARY

In this section, we prove that the Lipschitz continuity is the optimal regularity for ρ across the sonic boundary Γ_{sonic} , as well as at the intersection point P_1 between Γ_{sonic} and Γ_{shock} . In §4, we have shown that the solution ρ to the free boundary problem (4.15)–(4.19) is Lipschitz continuous in Ω up to the degenerate boundary Γ_{sonic} . Now we employ the approach introduced in Bae-Chen-Feldman [2] to analyze the finer behavior of ρ near the sonic circle $r = r_1 := c(\rho_1)$.

For $\varepsilon \in (0, \frac{c_1}{2})$, we denote by

$$\Omega_\varepsilon := \Omega \cap \{(r, \theta) : 0 < c_1 - r < \varepsilon\},$$

the ε -neighborhood of the sonic circle Γ_{sonic} within Ω .

In Ω_ε , we introduce the coordinates:

$$(5.1) \quad (x, y) = (c_1 - r, \theta - \theta_1).$$

One of our main observations is that it is convenient to study the regularity in terms of the difference between $c^2(\rho_1)$ and $c^2(\rho)$:

$$(5.2) \quad \psi := c^2(\rho_1) - c^2(\rho),$$

since ψ and ρ have the same regularity in Ω_ε .

It follows from (1.13) that ψ satisfies

$$(5.3) \quad \begin{aligned} \mathcal{L}_1 \psi &:= (2c_1 x - \psi + O_1) \psi_{xx} + (c_1 + O_2) \psi_x - (1 + O_3) \psi_x^2 \\ &\quad + (1 + O_4) \psi_{yy} - \left(\frac{1}{(\gamma-1)c_1^2} + O_5 \right) \psi_y^2 = 0 \quad \text{in } Q_{r,R}^+ \end{aligned}$$

in the (x, y) -coordinates, where

$$(5.4) \quad \begin{cases} O_1(x, \psi) = -x^2, \\ O_2(x, \psi) = -3x + \frac{\psi}{c_1}, \\ O_3(x, \psi) = -\frac{\gamma-2}{\gamma-1} (2c_1 x - \psi - x^2), \\ O_4(x, \psi) = \frac{c_1^2 - \psi}{(c_1 - x)^2} - 1, \\ O_5(x, \psi) = \frac{1}{(c_1 - x)^2} - \frac{1}{c_1^2}. \end{cases}$$

Moreover, ψ satisfies

$$(5.5) \quad \psi > 0 \quad \text{in } Q_{r,R}^+$$

and the following Dirichlet boundary condition:

$$(5.6) \quad \psi = 0 \quad \text{on } \partial Q_{r,R}^+ \cap \{x = 0\},$$

where

$$(5.7) \quad Q_{r,R}^+ := \{(x, y) : x \in (0, r), |y| < R\} \subset \mathbb{R}^2,$$

with $R = \theta_w - \theta_1$, since we can extend $\psi(x, y)$ from Ω_ε , by defining $\psi(x, y) = \psi(x, -y)$ for $(x, y) \in \Omega_\varepsilon$, and extend the domain Ω_ε with respect to y . Thus, without further comment, we study the behavior of ψ in $Q_{r,R}^+$.

It is easy to see that the terms $O_i(x, y)$, $i = 1, \dots, 5$, are continuously differentiable and

$$(5.8) \quad \frac{|O_1(x, y)|}{x^2}, \frac{|O_k(x, y)|}{x} \leq N \quad \text{for } k = 2, \dots, 5,$$

$$(5.9) \quad \frac{|DO_1(x, y)|}{x}, |DO_k(x, y)| \leq N \quad \text{for } k = 2, \dots, 5$$

in $\{x > 0\}$ for some constant N depending only on c_1 and γ . Inequalities (5.8) and (5.9) imply that the terms $O_i(x, y)$, $i = 1, \dots, 5$, are small. Thus, the main terms of (5.3) form the following equation:

$$(5.10) \quad (2c_1 - \psi) \psi_{xx} + c_1 \psi_x - \psi_x^2 + \psi_{yy} - \frac{1}{(\gamma-1)c_1^2} \psi_y^2 = 0 \quad \text{in } Q_{r,R}^+.$$

It follows from Lemmas 4.1 and 4.5 that

$$(5.11) \quad 0 \leq \psi \leq 2(c_1 - \vartheta)x,$$

where ϑ depends only on ρ_1 and γ . Then equation (5.10) is uniformly elliptic in every subdomain $\{x > \delta\}$ with $\delta > 0$. The same is true for (5.3) in $Q_{r,R}^+$ if r is sufficiently small.

Remark 5.1. If \hat{r} is sufficiently small, depending only on c_1 and γ , then (5.8)–(5.9), and (5.11) imply that (5.3) is uniformly elliptic with respect to ψ in $Q_{r,R}^+ \cap \{x > \delta\}$ for any $\delta \in (0, \frac{\hat{r}}{2})$. We will always assume such a choice of \hat{r} hereafter.

5.1. First-order lower bound of ψ . In order to prove that $C^{0,1}$ is the optimal regularity of ψ across the sonic boundary, our idea is to construct a positive sub-solution of (5.3), (5.5), and (5.6) first, which provides our desired lower bound of ψ .

Lemma 5.2. *Let ψ be a solution of the Dirichlet problem (5.3) and (5.5)–(5.6). Then there exist $\hat{r} > 0$ and $\mu > 0$, depending only on c_1 , γ , θ_w , and $\inf_{Q_{\hat{r},R}^+ \cap \{x > \hat{r}/2\}} \psi$,*

such that, for all $r \in (0, \frac{\hat{r}}{2}]$,

$$(5.12) \quad \psi(x, y) \geq \mu c_1 x \quad \text{in } Q_{r, \frac{15R}{16}}^+.$$

Proof. In the proof below, without further comment, all the constants depend only on the data, i.e., c_1 , \hat{r} , γ , θ_w , and $\inf_{Q_{\hat{r},R}^+ \cap \{x > \hat{r}/2\}} \psi$, unless otherwise is stated.

Fix y_0 with $|y_0| \leq \frac{15R}{16}$. We now prove that

$$(5.13) \quad \psi(x, y_0) \geq \frac{5}{8}\mu x \quad \text{for } x \in (0, r).$$

Without loss of generality, we may assume that $R = 2$ and $y_0 = 0$; otherwise, we set $\tilde{\psi}(x, y) = \psi(x, y_0 + \frac{R}{32}y)$ for all $(x, y) \in Q_{\hat{r}, 2}^+$. Then $\tilde{\psi}(x, y) \in C(\overline{Q_{\hat{r}, R}^+}) \cap C^2(Q_{\hat{r}, R}^+)$ satisfies (5.3) with (5.8)–(5.9) and (5.11) in $Q_{\hat{r}, 2}^+$, with some modified constants N , ϑ , and O_i , depending only on the corresponding quantities in the original equation and on R . Moreover,

$$\inf_{Q_{\hat{r}, 2}^+ \cap \{x > \hat{r}/2\}} \tilde{\psi} = \inf_{Q_{\hat{r}, R}^+ \cap \{x > \hat{r}/2\}} \psi.$$

Then (5.13) for ψ follows from (5.13) for $\tilde{\psi}$ with $y_0 = 0$ and $R = 2$. Thus we keep the original notation with $y_0 = 0$ and $R = 2$. That is, it suffices to prove that

$$(5.14) \quad \psi(x, 0) \geq \frac{5}{8}\mu x \quad \text{for } x \in (0, r).$$

By the Harnack inequality, we conclude that, for any $r \in (0, \frac{\hat{r}}{2})$, there exists $\sigma = \sigma(r) > 0$, depending only on r and the data c_1 , \hat{r} , γ , θ_w , and $\inf_{Q_{\hat{r}, R}^+ \cap \{x > \hat{r}/2\}} \psi$, such that

$$(5.15) \quad \psi \geq \sigma \quad \text{on } Q_{\hat{r}, 3/2}^+ \cap \{x > r\}.$$

Let $r \in (0, \frac{\hat{r}}{2})$ and

$$(5.16) \quad 0 < \mu_0 \leq \min\left\{\frac{\sigma(r)}{r}, c_1\right\},$$

where r will be chosen later. Define

$$(5.17) \quad g(y) = \begin{cases} \mu(y+1)^2, & -1 \leq y < -\frac{1}{2}, \\ \mu(2y^4 - 2y^2 + \frac{5}{8}), & -\frac{1}{2} \leq y \leq \frac{1}{2}, \\ \mu(y-1)^2, & \frac{1}{2} < y \leq 1. \end{cases}$$

Set $w(x, y) = \mu x g(y)$ with $g \in C^2([-1, 1])$. Then, using (5.16) and (5.17), we obtain that, for all $x \in (0, r)$ and $|y| < 1$,

$$(5.18) \quad \begin{cases} w(0, y) = 0 \leq \psi(0, y), \\ w(r, y) \leq \frac{5}{8}\mu r \leq \psi(r, y), \\ w(x, \pm 1) = 0 \leq \psi(x, \pm 1). \end{cases}$$

Therefore, we have

$$w \leq \psi \quad \text{on } \partial Q_{r,1}^+.$$

Next, we show that $w(x, y)$ is a strict subsolution $\mathcal{L}_1 w(x, y) > 0$ in $Q_{r,1}^+$, if the parameters are appropriately chosen. In fact,

$$\begin{aligned} & \mathcal{L}_1 w(x, y) \\ &= (c_1 g(y) - g^2(y)) \\ & \quad + x \left(g''(y) - \frac{1}{(\gamma-1)c_1^2} x (g'(y))^2 + \frac{O_2}{x} g(y) - \frac{O_3}{x} g^2(y) + O_4 g'(y) - x O_5 (g'(y))^2 \right). \end{aligned}$$

On one hand, for $1 - |y| < \varepsilon_0$ with ε_0 small enough, we can see

$$\begin{aligned} & g''(y) - \frac{1}{(\gamma-1)c_1^2} x (g'(y))^2 + \frac{O_2}{x} g(y) - \frac{O_3}{x} g^2(y) + O_4 g'(y) - x O_5 (g'(y))^2 \\ & \geq g''(y) - \frac{1}{(\gamma-1)c_1^2} x (g'(y))^2 - N x (g(y) + 1) g(y) - x N g'(y) + N x^2 (g'(y))^2 \\ & =: h(x, y). \end{aligned}$$

It is easy to see that $h(x, y)$ is continuous with respect to x , $h(0, y) = 0$, and that there exists $r_1 > 0$ such that $h(x, y) > 0$ for $r < r_1$.

On the other hand, for $1 - |y| > \varepsilon_0$,

$$\begin{aligned} & \mathcal{L}_1 w(x, y) \\ & \geq x \left(g''(y) - \frac{1}{(\gamma-1)c_1^2} x (g'(y))^2 + \frac{O_2}{x} g(y) - \frac{O_3}{x} g^2(y) + O_4 g'(y) - x O_5 (g'(y))^2 \right) \\ & \quad + \mu \varepsilon_0^2 (c_1 - \frac{5}{8}\mu). \end{aligned}$$

Then there exists $r_2 > 0$ such that the above inequality is positive.

We claim

$$\sup_{Q_{r,1}^+} (w - \psi) \leq \sup_{\partial Q_{r,1}^+} (w - \psi) \leq 0,$$

whenever $0 < r < r_0 := \min\{r_1, r_2\}$ and $\mu \in (0, \mu_0]$. Otherwise, there exists a point $(x_0, y_0) \in Q_{r,1}^+$ such that

$$\begin{aligned} 0 &< (\mathcal{L}_1 w - \mathcal{L}_1 \psi)(x_0, y_0) \\ &= (2c_1 x - \psi + O_1)(w - \psi)_{xx} + (c_1 + O_2)(w - \psi)_x - (1 + O_3)(w + \psi)_x(w - \psi)_x \\ &\quad + (1 + O_4)(w - \psi)_{yy} - \left(\frac{1}{(\gamma - 1)c_1^2} + O_5\right)(w + \psi)_y(w - \psi)_y \leq 0, \end{aligned}$$

where we have used the fact that $w_{xx} = 0$, which is a contradiction. Hence, we obtain our claim:

$$\psi(x, y) \geq w(x, y) = xf(y) \quad \text{in } Q_{r,1}^+.$$

In particular,

$$\psi(x, 0) \geq \frac{5}{8}\mu x \quad \text{for } x \in [0, r].$$

This implies (5.13). Then (5.12) holds by modifying μ , which is still denoted by μ . This completes the proof. \square

5.2. $C^{1,\alpha}$ -Estimate of ψ . If ψ satisfies (5.3), (5.5)–(5.6), and (5.11), it is expected that ψ is very close to $c_1 x$, which is a solution of (5.10). More precisely, we now prove

$$|\psi(x, y) - c_1 x| \leq Cx^{1+\alpha} \quad \text{for all } (x, y) \in Q_{\hat{r}, \frac{rR}{8}}^+$$

for some constant C .

To prove this, we study the function

$$(5.19) \quad W(x, y) := c_1 x - \psi(x, y).$$

By (5.3), W satisfies

$$\begin{aligned} (5.20) \quad \mathcal{L}_2 W &= (c_1 x + W + O_1)W_{xx} - (c_1 - O_2 - 2c_1 O_3)W_x + (1 - O_3)W_x^2 \\ &\quad + (1 + O_4)W_{yy} - \left(\frac{1}{(\gamma - 1)c_1^2} - O_5\right)W_y^2 \\ &= c_1 O_2 + c_1^2 O_3 \quad \text{in } Q_{\hat{r}, R}^+, \end{aligned}$$

$$(5.21) \quad W(0, y) = 0 \quad \text{on } \partial Q_{\hat{r}, R}^+ \cap \{x = 0\},$$

and

$$(5.22) \quad -(c_1 - \vartheta)x \leq W(x, y) \leq c_1 x \quad \text{in } Q_{\hat{r}, R}^+.$$

Then we establish the following two estimates.

Proposition 5.3. *Let c_1 , \hat{r} , R , and ϑ be the same as in Lemma 5.2. Then, for any $\alpha \in (0, 1)$, there exist positive constants r and A , which depend only on N , c_1 , \hat{r} , R , ϑ , and α , such that, if $W \in C(\overline{Q_{\hat{r}, R}^+}) \cap C^2(Q_{\hat{r}, R}^+)$ satisfies (5.20)–(5.22), then*

$$(5.23) \quad W(x, y) \leq Ax^{1+\alpha} \quad \text{in } Q_{r, \frac{3R}{4}}^+.$$

Proof. The main idea of the proof is the same as that in [2], and we only list the major procedure and the difference here. The complete proof is very long, which can be found in Appendix.

First, we prove that there exist $\alpha_1 \in (0, \frac{1}{2})$ and $r_1 > 0$ such that, if $W \in C(\overline{Q_{\hat{r}, R}^+}) \cap C^2(Q_{\hat{r}, R}^+)$ satisfies (5.20)–(5.22), then

$$W(x, y) \leq \frac{c_1(1 - \mu_1)}{r^\alpha} x^{1+\alpha} \quad \text{in } Q_{r, \frac{rR}{8}}^+,$$

whenever $\alpha \in (0, \alpha_1]$, $r \in (0, r_1]$, and $\mu_1 < \min\{\mu, \frac{1}{2}\}$, where μ is the constant determined by Lemma 5.2.

As in [2], we first note that, without loss of generality, we may assume that $R = 2$ and $y_0 = 0$. Then it suffices to prove that

$$W(x, 0) \leq \frac{c_1(1 - \mu_1)}{r^\alpha} x^{1+\alpha} \quad \text{for } x \in (0, r)$$

for some $r \in (0, r_0)$ and $\alpha \in (0, \alpha_1)$, under the assumptions that (5.20)–(5.22) hold in $Q_{\hat{r}, 2}^+$ and (A.25) holds in $Q_{r_0, 2}^+$.

For any given $r \in (0, r_0)$, let

$$\begin{aligned} A_1 r &= c_1(1 - \mu_1), \quad B_1 = c_1(1 - \mu_1), \\ v &= A_1 x^{1+\alpha}(1 - y^2) + B_1 xy^2. \end{aligned}$$

Then we obtain

$$W \leq v \quad \text{on } \partial Q_{r_0, 1}^+,$$

and

$$\mathcal{L}_2 v - \mathcal{L}_2 W - v_{xx}(v - W) < 0 \quad \text{in } Q_{r, 1}^+$$

whenever $r \in (0, r_1]$ and $\alpha \in (0, \alpha_1]$ so that

$$(5.24) \quad (2\alpha - 1)(\alpha + 1)c_1 A_1 < -\frac{\mu_1}{2},$$

and

$$(5.25) \quad r_1 < \min \left\{ \left(\frac{\mu_1}{4c_1} \right)^{\frac{1}{\alpha}}, \left(\frac{B_1 c_2 - B_1^2}{C} \right)^{\frac{1}{\alpha}}, r_0 \right\}.$$

Here we have used the fact that

$$v_{xx}(W - v) = \alpha(\alpha + 1)A_1 x^{\alpha-1}(1 - y^2)(W - v) \leq (1 + \alpha)\alpha c_1(1 - \mu_1)A_1 x^\alpha(1 - y^2).$$

Noting that

$$\begin{aligned} &\mathcal{L}_2 v - \mathcal{L}_2 W - v_{xx}(v - W) \\ &= (c_1 x + W + O_1)(v - W)_{xx} - (c_1 - O_2 - 2c_1 O_3)(v - W)_x \\ &\quad + (1 - O_3)(v + W)_x(v - W)_x + (1 + O_4)(v - W)_{yy} \\ &\quad - \left(\frac{1}{(\gamma - 1)c_1^2} - O_5 \right)(v + W)_x(v - W)_x, \end{aligned}$$

which implies

$$W \leq v \quad \text{in } Q_{r, 1}^+.$$

Next, we generalize the result for any $\alpha \in (0, 1)$, which suffices to show that for the case $\alpha > \alpha_1$.

Fix any $\alpha \in (\alpha_1, 1)$ and set the following comparison function:

$$v = \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}} x^{1+\alpha} (1 - y^2) + \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1}} x^{1+\alpha_1} y^2.$$

Then

$$(5.26) \quad W \leq v \quad \text{on } \partial Q_{r,1}^+ \quad \text{for } r \in (0, r_1].$$

As before, we can prove that

$$\mathcal{L}_2 v - \mathcal{L}_2 W - v_{xx}(v - W) < 0.$$

Then it is easy to prove that this proposition holds with

$$A = \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}}.$$

This completes the proof. \square

Proposition 5.4. *Let $c_1, \hat{r}, R, \vartheta$, and O_i be the same as in Lemma 5.2. Then, for any $\alpha \in (0, 1)$, there exist positive constants r and B , depending on $N, c_1, \hat{r}, R, \vartheta$, and α , so that, if $W \in C(\overline{Q_{\hat{r},R}^+}) \cap C^2(Q_{\hat{r},R}^+)$ satisfies (5.20)–(5.22), we have*

$$(5.27) \quad W(x, y) \geq -Bx^{1+\alpha} \quad \text{in } Q_{r, \frac{3R}{4}}^+.$$

Proof. Similar to the proof of Proposition 5.3, it suffices to prove that, with the assumption $R = 2$,

$$W(x, 0) \geq -\frac{c_1 - \vartheta}{r^\alpha} x^{1+\alpha} \quad \text{for } x \in (0, r)$$

for some $r > 0$ and $\alpha \in (0, \alpha_2)$. For this, we use the comparison function:

$$v(x, y) := -Lx^{1+\alpha}(1 - y^2) - Kxy^2, \quad \text{with } Lr^\alpha = K = \frac{c_1 - \vartheta}{r^\alpha}.$$

It is easy to check that

$$W \geq v \quad \text{on } \partial Q_{r,1}^+ \text{ for } r \in (0, r_1].$$

Then we follow the same procedure as in [2], except that $\mathcal{L}_2 v > \mathcal{L}_2 W$, to find that the conditions for the choice of $\alpha, r > 0$ are inequalities (5.24) and (5.25) with (μ_1, r_1) replaced by (β, r_2) , respectively, and with an appropriate constant C .

We claim that

$$\min_{Q_{r,1}^+} (W - v) \geq \min_{\partial Q_{r,1}^+} (W - v) \geq 0.$$

Otherwise, there exists a point $(x_0, y_0) \in Q_{r,1}^+$ such that $(W - v)(x_0, y_0) < 0$ and

$$\begin{aligned}
 0 &> (\mathcal{L}_2 W - \mathcal{L}_2 v)(x_0, y_0) \\
 &= (c_1 x + W + O_1)(W - v)_{xx} - (c_1 - O_2 - 2c_1 O_3)(W - v)_x \\
 &\quad + (1 - O_3)(W + v)_x(v - W)_x + (1 + O_4)(W - v)_{yy} \\
 &\quad - \left(\frac{1}{(\gamma-1)c_1^2} - O_5\right)(W + v)_x(W - v)_x + v_{xx}(W - v) \\
 &\geq 0 \quad \text{in } Q_{r,1}^+,
 \end{aligned} \tag{5.28}$$

which is a contradiction. This completes the proof for the case $\alpha \leq \alpha_2$.

For the case $\alpha \in (\alpha_2, 1)$, we set the comparison function:

$$u_-(x, y) := -\frac{c_1 - \vartheta}{r_2^{\alpha_2} r^{\alpha - \alpha_2}} x^{1+\alpha} (1 - y^2) - \frac{c_1 - \vartheta}{r_2^{\alpha_2}} x^{1+\alpha_2}.$$

Then, using the argument as before, we can choose $r > 0$ appropriately small such that

$$\mathcal{L}_2 u_- - \mathcal{L}_2 W > 0$$

holds for all $(x, y) \in Q_{r,1}^+$.

This completes the proof. \square

Lemma 5.5. *Let $\psi \in C(\overline{Q_{\hat{r},R}^+}) \cap C^2(Q_{\hat{r},R}^+)$ be a solution of the Dirichlet problem (5.3) and (5.5)–(5.6). Then $\psi \in C^{1,\alpha}(\overline{Q_{\hat{r}/2,R/2}^+})$ for any $\alpha \in (0, 1)$ with*

$$\psi_x(0, y) = c_1, \quad \psi_y(0, y) = 0 \quad \text{for any } |y| \leq \frac{R}{2}.$$

Proof. The proof is quite similar to that in [2], and the main difference is the scaling due to the different equations. Here we just list the main procedures. As in [2], we divide the proof into the following four steps.

Step 1. Let ψ be a solution of (5.3) in $Q_{\hat{r},R}^+$ for \hat{r} as in Remark 5.1. By Theorem 2.1, we have $\psi \in C^{1,\alpha}(\overline{Q_{r,R/2}^+})$ for some $r \in (0, \hat{r}/2)$. Thus, it suffices to show that, for any given $\alpha \in (0, 1)$, there exists $r > 0$ so that $\psi \in C^{1+\alpha}(\overline{Q_{r,R/2}^+})$ and

$$\psi_x(0, y) = c_1, \quad \psi_y(0, y) = 0 \quad \text{for all } |y| \leq \frac{R}{2}.$$

Let $W(x, y)$ be defined by (5.19). We need to show that, for any given $\alpha \in (0, 1)$, there exists $r > 0$ such that

- (i) $W \in C^{1,\alpha}(\overline{Q_{r,R/2}^+})$;
- (ii) $D(0, y) = 0$ for all $|y| \leq \frac{R}{2}$.

Step 2. By definition, W satisfies (5.20)–(5.22). For any given $\alpha \in (0, 1)$, there exists $r > 0$ so that (5.23) and (5.27) hold in $Q_{r,3R/4}^+$ by Propositions 5.3–5.4. Fix such $r > 0$.

Furthermore, since W satisfies estimate (5.22), we can introduce a cut-off function into the nonlinear term of (5.20). That is, we modify the nonlinear term away from the values determined by (5.22) to make the term bounded in $\frac{W}{x}$. Namely, fix $\zeta \in C^\infty(\mathbb{R})$ satisfying

$$(5.29) \quad \begin{aligned} & -(c_1 - \frac{\vartheta}{2})x \leq \zeta \leq (c_1 + \frac{\vartheta}{2})x \quad \text{on } \mathbb{R}, \\ & \zeta(s) = s \quad \text{on } (c_1 - \vartheta, c_1), \\ & \zeta(s) = 0 \quad \text{on } (2c_1 - \vartheta, c_1 + \beta). \end{aligned}$$

Then, from (3.26) and (3.28), it follows that W satisfies

$$(5.30) \quad \begin{aligned} & x(c_1 + \zeta(\frac{W}{x}) + O_1)W_{xx} - (c_1 - O_2 - 2c_1O_3)W_x + (1 - O_3)W_x^2 \\ & + (1 + O_4)W_{yy} - (\frac{1}{(\gamma-1)c_1^2} - O_5)W_y^2 \\ & = c_1O_2 + c_1^2O_3 \quad \text{in } Q_{\hat{r}, R}^+, \end{aligned}$$

Step 3. For $z := (x, y) \in Q_{r/2, R/2}^+$, define

$$R_z := \{(s, t) : |s - x| < \frac{x}{8}, |t - y| < \frac{\sqrt{x}}{8}\}.$$

Then

$$(5.31) \quad R_z \subset Q_{r, \frac{3R}{4}}^+ \quad \text{for any } z = (x, y) \in Q_{\frac{r}{2}, \frac{R}{2}}^+.$$

Fix $z_0 = (x_0, y_0) \in Q_{r/2, R/2}^+$. Rescaling W in R_{z_0} by defining

$$W^{(z_0)}(S, T) = \frac{1}{x_0^{1+\alpha}} W(x_0 + \frac{x_0}{8}S, y_0 + \frac{\sqrt{x_0}}{8}T) \quad \text{for } (S, T) \in Q_1,$$

where $Q_h = (-h, h)^2$ for $h > 0$. Then, as [2], $W^{(z_0)}$ satisfies the following equation for $(S, T) \in Q_1$:

$$(5.32) \quad \sum_{i=1}^2 A_{ii}(W^{(z_0)}, S, T) D_{ii}^2 W^{(z_0)} + \sum_{i=1}^2 B_i(DW^{(z_0)}, S, T) D_i W^{(z_0)} = F^{(z_0)} \quad \text{in } Q_1,$$

and, if $r > 0$ is sufficiently small, depending only on the data, then (5.32) is uniformly elliptic with elliptic constants depending on c_1 but independent of z_0 , and that the coefficients $A_{ii}(W^{(z_0)}, S, T)$, $B_i(DW^{(z_0)}, S, T)$, and $F^{(z_0)}(S, T)$, for $p \in \mathbb{R}^2$, $(S, T) \in \overline{Q_1}$, satisfy

$$\|W^{(z_0)}\|_{C^0(\overline{Q_1})} \leq \frac{c_1}{r^\alpha}, \quad \|A_{ii}\|_{C^0(\mathbb{R} \times \overline{Q_1})} \leq C, \quad \|(B_i, \frac{F^{(z_0)}}{r^{\frac{1}{2}-\beta}})\|_{C^0(\mathbb{R}^2 \times \overline{Q_1})} \leq C,$$

where C depends only on the data and is independent of z_0 . Then, by Theorem (A1) in [7],

$$(5.33) \quad \begin{aligned} \|W^{(z_0)}\|_{C^{1,\alpha}(\overline{Q_{1/2}})} & \leq C \left(\|W^{(z_0)}\|_{C^0(\overline{Q_1})} + \|F^{(z_0)}\|_{C^0(\overline{Q_1})} \right) \\ & \leq C \left(\frac{c_1}{r^\alpha} + r^{\frac{1}{2}-\beta} \right) =: \hat{C}, \end{aligned}$$

where C depends only on the data and α in this case. From (5.33), we have

$$(5.34) \quad |D_x W(x_0, y_0)| \leq C x_0^\alpha, \quad |D_y W(x_0, y_0)| \leq C x_0^{\alpha+\frac{1}{2}},$$

for all $(x_0, y_0) \in Q_{r/2, R/2}^+$.

Step 4. It remains to prove the C^α -continuity of DW in $\overline{Q_{r/2, R/2}^+}$.

For two distinct points $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in Q_{r/2, R/2}^+$ with $x_1 \leq x_2$. There are two cases:

Case 1: $z_1 \in R_{z_2}$. Then

$$x_1 = x_2 + \frac{x_2}{8}S, \quad y_1 = y_2 + \frac{\sqrt{x_2}}{8}T \quad \text{for some } (S, T) \in Q_1.$$

By (5.33), we have

$$\frac{|W_S^{(z_2)}(S, T) - W_S^{(z_2)}(0, 0)|}{|S^2 + T^2|^{\frac{\alpha}{2}}} \leq \hat{C}.$$

Since $x_2 \in (0, r)$ and $r \leq 1$, the last estimate implies

$$\frac{|W_x(x_1, y_1) - W_x(x_2, y_2)|}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{\frac{\alpha}{2}}} \leq \hat{C}.$$

Case 2: $z_1 \notin R_{z_2}$. Then either $|x_1 - x_2| \geq \frac{x_2}{8}$ or $|y_1 - y_2| > \frac{\sqrt{x_2}}{8}$. Since $0 \leq x_2 \leq r \leq 1$, we find

$$|z_1 - z_2|^\alpha > \left(\frac{x_2}{8}\right)^\alpha.$$

Thus, using (5.34) and $x_1 \leq x_2$, we obtain

$$\frac{|W_x(z_1) - W_x(z_2)|}{|z_1 - z_2|^\alpha} \leq \frac{|W_x(z_1)| + |W_x(z_2)|}{|z_1 - z_2|^\alpha} \leq \hat{C} \frac{x_1^\alpha + x_2^\alpha}{x_2^\alpha} \leq 2\hat{C}.$$

Since $z_1 \neq z_2$ are arbitrary points of $Q_{r/2, R/2}^+$, we obtain

$$[W_x]_{C^\alpha(\overline{Q_{r/2, R/2}^+})} \leq 2\hat{C},$$

where \hat{C} depends only on α , r , and the data. The estimate for W_y can be obtained similarly. In fact, for these derivatives, we obtain the stronger estimates: For any $\delta \in (0, \frac{r}{2}]$,

$$[W_y]_{C^\alpha(\overline{Q_{r/2, R/2}^+})} \leq \hat{C}\delta,$$

where \hat{C} depends only on α , r , and the data, but is independent of $\delta > 0$ and z_0 .

Thus, $W \in C^{1, \alpha}(\overline{Q_{r/2, R/2}^+})$ with $\|W\|_{C^{1, \alpha}(\overline{Q_{r/2, R/2}^+})}$ depending only on the data, because $r > 0$ depends on the data. Moreover, (5.34) implies that $DW(0, y) = 0$ for any $|y| \leq \frac{R}{2}$. This concludes the proof. \square

Theorem 5.6. *Let $\rho \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ be the solution of the free boundary problem (2.11)–(2.15) in §4. Then ρ cannot be C^1 across the degenerate sonic boundary Γ_{sonic} .*

Proof. On the contrary, assume that ρ is C^1 across the degenerate sonic boundary Γ_{sonic} . Then $\psi = c_1^2 - \rho^{\gamma-1}$ is also C^1 across Γ_{sonic} , satisfying the Dirichlet boundary problem (5.3) and (5.5)–(5.6) in $Q_{r,R}^+$ for some small $r > 0$. Moreover, since $\psi \equiv 0$ in the region $P_1 P_2 O P_3$, we have $D\psi = 0$ at all $(\xi, \eta) \in \Gamma_{\text{sonic}}$.

Let $(0, y_0)$ be a point in the relative interior of Γ_{sonic} . Then $(0, y_0) + Q_{r,R}^+ \subset \Omega_{\varepsilon_0}$ if $r, R > 0$ are sufficiently small. By shifting the coordinates $(x, y) \rightarrow (x, y - y_0)$, we can assume $(0, y_0) = (0, 0)$ and $Q_{r,R}^+ \subset \Omega_{\varepsilon_0}$. Note that the shifting coordinates in the y -direction does not change the expression in (5.3).

Since $\psi \in C^1(\Omega_{\varepsilon_0} \cup \Gamma_{\text{sonic}})$ with $D\psi = 0$ on Γ_{sonic} , reducing r if necessary, we obtain

$$|\psi| \leq \delta x,$$

where $\delta > 0$ is so small that (5.11) holds in $Q_{r,R}^+$. Also, we have $\psi = c_1^2 - c^2(\rho) > 0$ in $Q_{r,R}^+$. Now we can apply Lemma 5.2 to conclude

$$\psi(x, y) \geq \mu c_2 x \quad \text{on } Q_{r, \frac{15R}{16}}^+$$

for some $\mu, r > 0$. This contradicts the fact that $D\psi(0, y) = 0$ for all $y \in (-R, R)$, that is, $D\psi(\xi, \eta) = 0$ at any $(\xi, \eta) \in \Gamma_{\text{sonic}}$. This implies that ρ cannot be C^1 across the degenerate sonic boundary Γ_{sonic} . \square

We now study more detailed regularity of ρ near the sonic circle. From now on, we use a localized version of Ω_ε : For given neighborhood $\mathcal{N}(\Gamma_{\text{sonic}})$ of Γ_{sonic} and $\varepsilon > 0$, define

$$\Omega_\varepsilon := \Omega \cap \mathcal{N}(\Gamma_{\text{sonic}}) \cap \{x < \varepsilon\}.$$

Since $\mathcal{N}(\Gamma_{\text{sonic}})$ is fixed in the following theorem, we do not specify the dependence of Ω_ε on $\mathcal{N}(\Gamma_{\text{sonic}})$.

We first show the regularity part of Theorem 2.1.

Theorem 5.7. *Let ρ be the solution of the free boundary problem (4.15)–(4.19) established in §4 and satisfy the properties: There exists a neighborhood $\mathcal{N}(\Gamma_{\text{sonic}})$ of Γ_{sonic} such that, for $\psi := c_1^2 - c^2(\rho)$,*

(a) *ψ is $C^{0,1}$ across the part Γ_{sonic} of the degenerate sonic boundary;*

(b) *there exists $\vartheta_0 > 0$ so that, in the coordinates (5.1),*

$$(5.35) \quad |\psi| \leq (2c_1 - \vartheta_0)x \quad \text{in } \Omega \cap \mathcal{N}(\Gamma_{\text{sonic}}).$$

Then we have

(i) *There exists $\varepsilon_0 > 0$ such that ψ is $C^{1,\alpha}$ in Ω up to Γ_{sonic} away from the point P_1 for any $\alpha \in (0, 1)$. That is, for any $\alpha \in (0, 1)$ and any $(\xi_0, \eta_0) \in$*

$\overline{\Gamma_{\text{sonic}}} \setminus P_1$, there exists $K < \infty$ depending only on $\rho_0, \rho_1, \gamma, \varepsilon_0, \alpha, \|\psi\|_{C^{0,1}}$, and $d = \text{dist}((\xi_0, \eta_0), \Gamma_{\text{sonic}})$ so that

$$\|\psi\|_{1, \alpha; \overline{B_{d/2}}(\xi_0, \eta_0) \cap \Omega_{\varepsilon_0/2}} \leq K;$$

(ii) For any $(\xi_0, \eta_0) \in \Gamma_{\text{sonic}} \setminus P_1$,

$$\lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Omega}} D_r \psi = c_1;$$

(iii) The limit $\lim_{\substack{(\xi, \eta) \rightarrow P_1 \\ (\xi, \eta) \in \Omega}} D_r \psi$ does not exist.

The proof is quite similar to the one in [2], which can be achieved by following the proof of Theorem 4.2 in [2] step by step. Hence we omit the proof here.

6. PROOF OF THEOREM 2.1: GLOBAL SOLUTIONS

Finally, we show that the solution established above is a global solution indeed, valid through the sonic circle Γ_{sonic} , as claimed in Theorem 2.1.

Since ρ is only Lipschitz continuous across the sonic circle, we treat the solution in the weak sense: For every $\zeta \in C_c^\infty(\Omega_-)$, with Ω_- denoting the region of the left state,

$$\int_{\Omega_-} ((c^2 - r^2)\rho_r \zeta_r + \frac{c^2}{r^2}\rho_\theta \zeta_\theta - \frac{c^2}{r}\rho_r \zeta) \, dr d\theta = 0.$$

Notice that ρ is Lipschitz continuous across the sonic circle. Then, due to the Green theorem, the integrand is equal to 0 if and only if

$$(6.1) \quad [((c^2 - r^2)\rho_r, \frac{c^2}{r^2}\rho_\theta) \cdot \nu] = 0 \quad \text{on } \Gamma_{\text{sonic}},$$

where the bracket $[\cdot]$ denotes the difference of the quantity between two sides of the sonic circle, and ν is the normal direction. It is obvious because from the facts that $(\rho_r, \rho_\theta) = (-c_1, 0)$ up to the sonic circle from the subsonic domain obtained in Lemma 5.5, $(\rho_r, \rho_\theta) = (0, 0)$ from the supersonic domain and the fact that $c^2 - r^2 = 0$ on the sonic circle.

This completes the proof of Theorem 2.1.

7. EXISTENCE AND REGULARITY OF GLOBAL SOLUTIONS OF THE NONLINEAR WAVE SYSTEM

In our main theorem, Theorem 2.1, we have constructed a global solution ρ of the second-order partial differential equation (4.15) in Ω , combining this function with $\rho = \rho_1$ in state (1) and $\rho = \rho_0$ in state (0). That is, we have obtained the global density function ρ that is piecewise constant in the supersonic region, which is Lipschitz continuous across the degenerate sonic boundary Γ_{sonic} from Ω to state (1).

To recover the momentum components, m and n , we can integrate the second and the third equation in (1.7), which are equivalent to the equations in (2.8) in the polar coordinates. These can be also written in the radial variable r ,

$$(7.1) \quad \frac{\partial m}{\partial r} = \frac{1}{r}p(\rho)_\xi, \quad \frac{\partial n}{\partial r} = \frac{1}{r}p(\rho)_\eta,$$

and integrated from the boundary of the subsonic region toward the origin.

Note that we have proved that the limit of $D\rho$ does not exist at P_1 as (ξ, η) in Ω tends to (ξ_1, η_1) , but $|Dc(\rho)|$ has a upper bound. Thus, $p(\rho)$ is Lipschitz, which implies that (m, n) are at least Lipschitz across the sonic circle Γ_{sonic} .

Furthermore, (m, n) have the same regularity as ρ inside Ω except the origin $r = 0$. However, (m, n) may be multi-valued at the origin $r = 0$.

In conclusion, we have

Theorem 7.1. *Let the wedge angle θ_w be between $-\pi$ and 0. Then there exists a global solution $(\rho, m, n)(r, \theta)$ with the free boundary $r = r(\theta)$, $\theta \in [\theta_w, \theta_1]$, of Problem 2 such that*

$$(\rho, m, n) \in C^{2+\alpha}(\Omega), \quad \rho \in C^\alpha(\bar{\Omega}), \quad r \in C^{2+\alpha}([\theta_w, \theta_1]) \cap C^{1,1}([\theta_w, \theta_1]),$$

and $(\rho, m, n) = (\rho_1, m_1, 0)$ in the domain $\{\xi < \xi_1, r > r_1\}$ and $(\rho_0, 0, 0)$ in the domain $\{\xi > \xi_1, \eta > \eta_1\} \cup \{r > r(\theta), \theta \in [\theta_w, \theta_1]\}$. Moreover, the solution $(\rho, m, n)(r, \theta)$ with the free boundary $r = r(\theta)$ satisfies the following properties:

- (i) $\rho > \rho_0$ on the shock Γ_{shock} , that is, the shock Γ_{shock} is separated from the sonic circle C_0 of state (0);
- (ii) The shock Γ_{shock} is convex in the self-similar coordinates (ξ, η) ;
- (iii) The solution (ρ, m, n) is $C^{1,\alpha}$ up to Γ_{sonic} and Lipschitz continuous across Γ_{sonic} ;
- (iv) The Lipschitz regularity of the solution across Γ_{sonic} and at P_1 from the inside is optimal;
- (v) The momentum components (m, n) may be multi-valued at the origin.

APPENDIX A. PROOFS OF LEMMAS 3.4, 3.8, 3.9, AND PROPOSITION 5.3

In this appendix, we give detailed proofs of Lemmas 3.4, 3.8, and 3.9, and Proposition 5.3 to make the arguments self-contained. The proofs here are slightly different from the ones in [2], [5], and [18], respectively, in order to be adapted for our problem, but the main ideas are the similar.

A.1. Proof of Lemma 3.4. Away from a neighborhood $B_{d_0}(P_2)$ of P_2 , the operator M is oblique. Thus we can apply Theorem 6.30 in [16] to obtain (3.17) in $\Gamma(d) \setminus \{B_{d_0}(P_1) \cup B_{d_0}(P_2)\}$, with a constant C depending on ε , α_1 , Ω , d_0 , and K_0 . Hence we consider only the estimates near P_2 in the rest of the proof.

For a given solution u to (3.12)–(3.13), we define

$$(A.1) \quad v = \frac{u}{1 + \|Du\|_0} \quad \text{and} \quad z = Mv = \sum_{i=1}^2 \beta_i(P) D_i v.$$

For $d_0 > 0$ small enough, $O \notin B_{d_0}(P_2)$.

Let ζ be the regularized distance function (from the boundary component Γ_{shock}), which has the properties:

$$1 \leq \frac{\zeta}{d} \leq 2, \quad 0 < \zeta_0 \leq |D\zeta| \leq \zeta_D, \quad |D^2\zeta| \leq \zeta_D d^{\alpha_1-1}$$

for two positive constants ζ_0 and ζ_D (see [20]). A smooth approximation to $d(P) = \text{dist}(P, \Gamma_{\text{shock}})$ is necessary since Γ_{shock} has minimal regularity.

We first construct barrier functions $\pm g$ for z on $B := B_{d_0}(P_2) \cap \overline{\Omega}$, by finding a suitable positive, increasing function g , $g(0) = 0$, such that

$$|z| \leq g.$$

We now derive some estimates for $z \pm f$ hereafter.

First, we have

$$D_i(z + g) = \sum_{j=1}^2 (\beta_j D_{ij} v + D_i \beta_j D_j v) + g' \zeta_i,$$

that is,

$$(A.2) \quad \sum_{j=1}^2 \beta_j D_{ij} v = D_i(z + g) - \left(\sum_{j=1}^2 D_i \beta_j D_j v + g' \zeta_i \right).$$

We also have

$$(A.3) \quad \begin{aligned} D_{ij}(z + g) &= \sum_{k=1}^2 (\beta_k D_{ijk} v + D_j \beta_k D_{ik} v + D_i \beta_k D_{jk} v + D_{ij} \beta_k D_k v) \\ &\quad + g' \zeta_{ij} + g'' \zeta_i \zeta_j. \end{aligned}$$

In addition, since w satisfies (W2) in Definition 3.2 with a given constant K , we obtain the following estimates on the derivatives of a_{ij} :

$$(A.4) \quad |D_{(r,\theta)}(a_{ij}^\varepsilon)| \leq |a_{ij,x}^\varepsilon| + |a_{ij,u}^\varepsilon| |D_{(r,\theta)} w| \leq |a_{ij,x}^\varepsilon| + |a_{ij,u}^\varepsilon| \|w\|_1^{-\gamma_1} d^{\gamma_1-1} \leq m d^{\gamma_1-1},$$

$$(A.5) \quad \begin{aligned} |D_{(r,\theta)}^2(a_{ij}^\varepsilon)| &\leq |a_{ij,x,x}^\varepsilon| + 2|a_{ij,x,u}^\varepsilon| |D_{(r,\theta)} w| + |a_{ij,u,u}^\varepsilon| |D_{(r,\theta)} w|^2 + |a_{ij,u}^\varepsilon| |D_{(r,\theta)}^2 w| \\ &\leq |a_{ij,x,x}^\varepsilon| + 2|a_{ij,x,u}^\varepsilon| \|w\|_1^{-\gamma_1} d^{\gamma_1-1} + |a_{ij,u,u}^\varepsilon| (\|w\|_1^{-\gamma_1} d^{\gamma_1-1})^2 \\ &\quad + |a_{ij,u}^\varepsilon| \|w\|_2^{-\gamma_1} d^{\gamma_1-2} \\ &\leq m(d^{\gamma_1-2} + d^{2\gamma_1-2}). \end{aligned}$$

Here the subscripts denote the partial derivatives of $a_{ij}^\varepsilon(r, \theta)$ as functions of (x, u) . The symbol $m = m(K)$ denotes a quantity that depends on the structure of the

derivative of a_{ij}^ε and the bound K on w . We absorb the terms which are less singular as $d \rightarrow 0$. In a similar way, we obtain the estimates on the derivatives of $\beta = (\beta_1, \beta_2)$.

Let $\gamma_2 = \min\{\gamma_1, \alpha_1\}$. Then,

$$(A.6) \quad |D\beta_i| \leq md^{\gamma_2-1}, \quad |D^2\beta_i| \leq m(d^{\gamma_2-2} + d^{2\gamma_2-2}),$$

where $m = m(K) > 0$ depends on the structure of the derivatives of β . Here, we have used the fact that r' , r'' , and r''' are bounded by d^{α_1} , d^{α_1-1} , and d^{α_1-2} , respectively, as we can apply Lemma 2.8 of [15] to $r(\theta)$.

Since $\beta_1(P_2, w) = 0$ and $\beta_2(P_2, w) \neq 0$, by continuity, we have $\beta_2(P_2, w) \neq 0$ in $B_{d_0}(P_2)$, $0 < d_0 < d_1$, with d_1 depending on $\|w\|_{C^1}$ and $\|r\|_{C^1}$, which is small enough. Now we solve the two equations in (A.2) along with $Lv = 0$, i.e.,

$$(A.7) \quad a_{ij}^\varepsilon D_{ij}v = -(D_j a_{ij}^\varepsilon D_i v + b_i^\varepsilon D_i v),$$

as a linear system for the three derivatives $D_{ij}v$.

The assumption that β_2 is bounded away from zero, coupled with the ellipticity of L , gives a uniform bound $c_1(\Lambda, \lambda, \|\beta\|_0)$ on the inverse of the coefficient matrix of the linear system. Here we may let Λ and λ be the eigenvalues of a_{ij}^ε restricted to B , which are bigger than a positive constant depending only on the data and $\delta > 0$. Furthermore, we can use (A.4) and (A.6) to estimate the right-hand sides of (A.7) respectively. We obtain

$$(A.8) \quad |D^2v| \leq c_1(\Lambda, \lambda, \|\beta_2^{-1}\|_0)(|D(z+f)| + (md^{\gamma_2-1} + \|b^\varepsilon\|_0)|Dv| + f'\zeta_D).$$

This controls the second derivatives of v in terms of $|D(z+f)|$.

Now we proceed to obtain the bounds for $z+f$. The idea is to find a second-order elliptic operator, or $z+g > 0$, in B involving the third derivative of v and simultaneously to force $z+g > 0$ on ∂B , by choice of the function g . We estimate these first. Using $Lv = 0$, $|Dv| \leq 1$, and (A.5), we have

$$\begin{aligned} a_{ij}^\varepsilon D_{ijk}v &= -(D_k a_{ij}^\varepsilon D_{ij}v + D_j a_{ik}^\varepsilon D_{ij}v + b_i^\varepsilon D_{ik}v + D_{jk} a_{ij}^\varepsilon D_i v + D_k b_i^\varepsilon D_i v) \\ &\leq (md^{\gamma_2-1} + \|b^\varepsilon\|_0)|D^2v| + (md^{\gamma_2-2} + md^{2\gamma_2-2} + \|b^\varepsilon\|_1)|Dv| \\ &\leq C_1(md^{\gamma_2-1} + \|b^\varepsilon\|_0)|D(z+g)| + c_1(md^{\gamma_2-1} + \|b^\varepsilon\|_0)^2 \\ &\quad + C_1(md^{\gamma_2-1} + \|b^\varepsilon\|_0)g'\zeta_D + md^{\gamma_2-2} + md^{2\gamma_2-2} + \|b^\varepsilon\|_1 \\ &\leq C_2\left((md^{\gamma_2-1} + 1)|D(z+g)| + (md^{\gamma_2-1} + 1)^2 \right. \\ &\quad \left. + (md^{\gamma_2-1} + 1)g' + md^{\gamma_2-2} + md^{2\gamma_2-2} + 1\right), \end{aligned}$$

where $C_2 = c_2(\Lambda, \lambda, \rho_2, \|\beta_2^{-1}\|_0, \|b\|_1, \zeta_D)$. Thus, using (A.7) and making the estimates indicated, we have

$$\begin{aligned}
(A.9) \quad & a_{ij}^\varepsilon D_{ij}(z+g) \\
& \leq C_2 \|\beta\|_0 \left((md^{\gamma_2-1} + 1) |D(z+g)| + (md^{\gamma_2-1} + 1)^2 \right. \\
& \quad \left. + (md^{\gamma_2-1} + 1)g' + md^{\gamma_2-2} + md^{2\gamma_2-2} + 1 \right) \\
& \quad + 2\Lambda md^{\gamma_2-1} c_1 \left(|D(z+g)| + md^{\gamma_2-1} + \|b\|_0 + g'\zeta_D \right) \\
& \quad + \Lambda m(d^{\gamma_2-2} + d^{2\gamma_2-2}) + \Lambda g' |\zeta_{ij}| + g'' \tilde{a}_{ij} \zeta_i \zeta_j \\
& \leq C_3 \left((md^{\gamma_2-1} + 1) |D(z+g)| + md^{\gamma_2-2} + (m^2 + m)d^{2\gamma_2-2} + md^{\gamma_2-1}(1+g') \right) \\
& \quad + \Lambda g' |\zeta_{ij}| + g'' \tilde{a}_{ij} \zeta_i \zeta_j.
\end{aligned}$$

Here $C_3 > 0$ is a constant depending on the same parameters as C_1 and C_2 , and the terms that are bounded as $d \rightarrow 0$ have again been omitted.

Now we define

$$(A.10) \quad L_1(z+g) = a_{ij}^\varepsilon D_{ij}(z+g) - c_3(md^{\gamma_2-1} + 1) |D(z+g)|.$$

Then we have

$$L_1(z+g) \leq C_3 \left(md^{\gamma_2-2} + (m^2 + m)d^{2\gamma_2-2} + md^{\gamma_2-1}(1+g') \right) + \Lambda g' |\zeta_{ij}| + g'' a_{ij}^\varepsilon \zeta_i \zeta_j.$$

To obtain this estimate, we have assumed that $g'' < 0$ and estimated that

$$(A.11) \quad g'' a_{ij}^\varepsilon \zeta_i \zeta_j \leq g'' \min(a_{ij}^\varepsilon \zeta_i \zeta_j) \leq g'' \lambda |D\zeta|^2 \leq g'' \lambda \zeta_0^2,$$

where we have used the fact that $|D\zeta| \geq \zeta_0$.

Now we specify $g(\zeta) = g_0 \zeta^\mu$ for any $\mu < \gamma_2$. Thus, we have

$$g'' = g_0 \mu(\mu - 1) \zeta^{\mu-2} \leq 2^{\mu-2} g_0 \mu(\mu - 1) d^{\mu-2} < 0,$$

and

$$g' d^{\alpha_1-1} \leq g_0 d^{\mu+\alpha_1-2}.$$

We choose g_0 large enough and $d_2 \in (0, 1)$ small enough to obtain

$$L_1(z+g) < 0 \quad \text{in } B_{d_0}$$

for every $d_0 \leq d_2$. We define $d_0 := \min\{d_1, d_2\}$.

Additionally, (3.17) is valid on ∂B since it holds near Γ_{shock} and away from P_1 and P_2 . We can choose f_0 larger if necessary so that $(z+g) > 0$ on ∂B . Therefore, by the maximum principle, $z+g > 0$ in B . Thus, $z > -g$ in B .

Similarly, for $z - g(\zeta)$, it is easy to check that

$$\begin{aligned}
& a_{ij}^\varepsilon D_{ij}(z-g) \\
& \geq -C_3 \left((md^{\gamma_2-1} + 1) |D(z-g)| + md^{\gamma_2-2} + (m^2 + m)d^{2\gamma_2-2} + md^{\gamma_2-1}(1+g') \right) \\
& \quad - \Lambda g' |\zeta_{ij}| - g'' a_{ij}^\varepsilon \zeta_i \zeta_j.
\end{aligned}$$

We define

$$(A.12) \quad L_2(z - g) := a_{ij}^\varepsilon D_{ij}(z - g) + c_3(md^{\gamma_2-1} + 1)|D(z - g)|.$$

By a similar argument, we have $L_2\psi_2 > 0$ in B . Therefore, we have

$$z < g \quad \text{in } B.$$

Let $F := L_1(z + g) \leq Cd^{\mu-2}$. It follows from (A.10) that

$$(A.13) \quad \begin{aligned} a_{ij}^\varepsilon D_{ij}(z + g) &= C_3(md^{\gamma_2-1} + 1)|D(z + g)| + F \\ &\leq C_3(md^{\gamma_2-1} + 1)|D(z + g)| + Cd^{\mu-2}. \end{aligned}$$

Moreover, we can use the Schauder estimates (cf. Lemma 6.20 in [16]) to obtain

$$\begin{aligned} \|z + g\|_{2+\gamma_2, B}^{(-\mu)} &\leq C(\|z + g\|_{0, B}^{(-\mu)} + \|F\|_{0, \gamma_2, B}^{(2-\mu)}) \\ &\leq C(md^{\gamma_2-1} + 1)|D(z + g)| + Cd^{\mu-2}. \end{aligned}$$

Using the interpolation inequality, we have

$$\|z + g\|_{2+\gamma_2, B}^{(-\mu)} \leq C(m).$$

Finally, this leads to

$$(A.14) \quad |D(z + g)| \leq \|(z + g)\|_{1+\gamma_2}^{(1-\mu)} d^{\mu-1} \leq C(m)d^{\mu-1} \quad \text{for } d < d_0.$$

We now use (A.14) in (A.8) and drop the lower-order terms to obtain

$$|D^2v| \leq c_1(|D\psi_1| + md^{\gamma_2-1} + g') \leq Cd^{\mu-1}.$$

Using Lemma 2.1 in [15], we have

$$\|Dv\|_\mu = \|Dv\|_\mu^{-\mu} \leq C(\mu)\|Dv\|_1^{-\mu},$$

which leads to

$$\|v\|_{1+\mu} \leq C.$$

Finally, using the definition of v in (A.1), we apply the interpolation inequality with small $\vartheta > 0$ to obtain

$$(A.15) \quad \|u\|_{1+\mu} \leq C(1 + \|Du\|_0) \leq C(1 + \vartheta\|u\|_{1+\mu} + C_\vartheta\|u\|_0)$$

and thus (3.17) holds. Therefore, we obtain the Hölder gradient estimate at Γ_{shock} for the solution u of (3.12). This completes the proof.

A.2. Proof of Lemma 3.8. For the notational simplicity, throughout the proof, we write $\rho = \rho^{\varepsilon, \delta}$.

We show our claim by contradiction. More precisely, assume that there exists a non-empty set $B = \{X \in \overline{\Gamma_{\text{shock}}} : \bar{c}(\rho, \rho_0) - r > 0\}$ and a point $X \in B$ such that

$$\max_{\overline{B}} (\bar{c}^2(\rho, \rho_0) - r^2) = (\bar{c}^2 - r^2)(X) = m > 0.$$

It is clear that $X \neq P_1, P_2$. Therefore, if X exists, then $X \in \Gamma_{\text{shock}} \setminus \{P_1, P_2\}$. Then X can be either a local maximum point or a saddle point in $\Omega \cup \Gamma_{\text{shock}}$. We show that both cases can not occur, which implies that such X does not exist.

First, if X is a local maximum point, then the tangential derivative becomes

$$r'((\bar{c}^2)'\rho_r - 2r) + (\bar{c}^2)'\rho_\theta = 0,$$

while the normal derivative at X becomes

$$(\bar{c}^2)'\rho_r - 2r - r'(\bar{c}^2)'\rho_\theta \geq 0,$$

which leads to

$$(A.16) \quad ((\bar{c}^2)'\rho_r - 2r)(1 + (r')^2) \geq 0 \quad \text{at } X.$$

On the other hand, multiplying \bar{c}' both sides of $M\rho = 0$ and using the equation due to the tangential derivative at X , it follows from (A.16) that

$$\begin{aligned} 0 &= (\bar{c}^2)' \sum_{i=1}^2 \beta_i D_i \rho \\ &= \beta_1 (\bar{c}^2)'\rho_r + \beta_2 \delta_1 r' (- (\bar{c}^2)'\rho_r + 2r) \\ &= (\bar{c}^2)'\rho_r (\beta_1 - \beta_2 r') + 2r r' \beta_2 \\ &= -2(\bar{c}^2)'\rho_r r' (c^2 - \bar{c}^2) r^2 + 2r r' (3c^2(r^2 - \bar{c}^2) - \bar{c}^2(c^2 - r^2)) \\ &\leq -4r^3 r' (c^2 - \bar{c}^2) + 6r r' c^2 (r^2 - \bar{c}^2) < 0, \end{aligned}$$

since we have $r'(\theta) > 0$. This is a contradiction.

Next, if X is a saddle point, then multiplying $(\bar{c}^2)'$ both sides of $Q^\varepsilon \rho = 0$ yields

$$\begin{aligned} 0 &= (\bar{c}^2)' Q^\varepsilon \rho \\ &= \sum_{i=1}^2 a_{ii}^\varepsilon \left(D_{ii}(\bar{c}^2) - \frac{(\bar{c}^2)''}{((\bar{c}^2)')^2} |D_i \bar{c}^2|^2 \right) + \frac{a}{(\bar{c}^2)'} \left((\bar{c}^2)_r^2 + \frac{1}{r^2} (\bar{c}^2)_\theta^2 \right) + \tilde{b}^\varepsilon (\bar{c}^2)_r, \end{aligned}$$

where $a = \frac{dc^2}{d\rho}$ and $\tilde{b}^\varepsilon = b^\varepsilon - 2r$. Thus we can write

$$L\bar{c}^2 = \sum_{i=1}^2 a_{ii}^\varepsilon D_{ii}(\bar{c}^2) + a_1 (\bar{c}^2)_r^2 + a_2 (\bar{c}^2)_\theta^2 + \tilde{b}^\varepsilon (\bar{c}^2)_r,$$

where $a_1 = -a_{11}^\varepsilon + \frac{a}{(\bar{c}^2)'} and $a_2 = -a_{22}^\varepsilon + \frac{a}{r^2(\bar{c}^2)'}$.$

Since X is a saddle point, we can construct a barrier function ψ so that $X = (r_x, \theta_x)$ is a maximum point along the normal direction.

We define $d := r_x - r + r'(\theta_x)(\theta - \theta_x)$ and a set

$$W := \{(r, \theta) \in \Omega : d > 0\} \cap \{(r, \theta) \in \Omega : \bar{c}^2 - r^2 > m\}.$$

Note that, since X is a saddle point, there are interior points of the set W , especially in the inward normal direction. We note that, since $\bar{c}^2(\rho, \rho_0) > r^2 + m$ in W and $c(\rho) > \bar{c}(\rho, \rho_0)$ for $\rho > \rho_0$, there exists a constant $e_0 > 0$ such that

$$c^2(\rho) - r^2 \geq e_0 > 0 \quad \text{in } \overline{W}.$$

Set $u = \bar{c}^2 - r^2 - m$. Then

$$\begin{aligned}
0 &= \sum_{i=1}^2 (a_{ii}^\varepsilon D_{ii}(\bar{c}^2) + a_i |D_i(\bar{c}^2)|^2) + b(\bar{c}^2)_r \\
&= \sum_{i=1}^2 (a_{ii}^\varepsilon D_{ii}(u + r^2 + m) + a_i |D_i(u + r^2 + m)|^2) + b(u + r^2 + m)_r \\
&= \sum_{i=1}^2 (a_{ii}^\varepsilon D_{ii}u + a_i |D_iu|^2) + (4ra_1 + \tilde{b}^\varepsilon)u_r + 2a_{11}^\varepsilon + 4r^2a_1 + 2r\tilde{b}^\varepsilon,
\end{aligned}$$

where $b_0 := 4a_1r + \tilde{b}^\varepsilon$ and $f_0 := 2a_{11}^\varepsilon + 4r^2a_1 + 2r\tilde{b}^\varepsilon$. We now let

$$w := \frac{1}{\mu_0}(e^{\mu_0 u} - 1), \quad \mu_0 > 0.$$

Then we have

$$\begin{aligned}
0 &= \sum_{i=1}^2 (a_{ii}^\varepsilon D_{ii}u + a_i |D_iu|^2) + b_0u_r + f_0 \\
&= e^{-\mu_0 u} \sum_{i=1}^2 (a_{ii}^\varepsilon (D_{ii}w - \mu_0 e^{-\mu_0 u} |D_iw|^2) + a_i e^{-2\mu_0 u} |D_iw|^2) + e^{-\mu_0 u} b_0w_r + f_0.
\end{aligned}$$

Hence, we have

$$\sum_{i=1}^2 a_{ii}^\varepsilon D_{ii}w + b_0w_r + e^{\mu_0 u} f_0 \geq e^{-\mu_0 u} \sum_{i=1}^2 (a_{ii}^\varepsilon \mu_0 - a_i) |D_iw|^2 \geq 0,$$

by choosing $\mu_0 = \max\{a_i\}/e_0$, where $a_{11}^\varepsilon, a_{22}^\varepsilon \geq e_0 > 0$ in W , and μ_0 and e_0 are independent of ε .

Thus, we find $\psi(d)$ with $\psi' \geq 0$ and $\psi'' < 0$ satisfying

$$L_1\psi = \sum_{i=1}^2 a_{ii} D_{ii}w + b_0w_r + e^{\mu_0 u} f_0 \leq e_0\psi'' + b_1\psi' + f_1 \leq 0,$$

where $b_1 = \max_W(4a_1r + b)$, $f_1 = \max_W e^{\mu_0 u} (2a_{11}^\varepsilon + 4r^2a_1 + 2r\tilde{b}^\varepsilon)^+$, and $\varepsilon \leq \varepsilon_0$. The solution to the equation:

$$e_0\psi'' + b_1\psi' + f_1 = 0$$

is

$$\psi = \frac{m_0b_1 + d_1f_1}{b_1} \frac{1 - e^{-b_1d/e_0}}{1 - e^{-b_1d_1/e_0}} - \frac{f_1}{b_1}d,$$

which satisfies the boundary condition:

$$\psi(0) = 0, \quad \psi(d_1) = m_0,$$

with $m_0 = \frac{e^{\mu_0 u_{\max}} - 1}{\mu_0}$, where $u_{\max} = \max_W u = \max_W(\bar{c}^2 - r^2 - m)$ and $d_1 > 0$.

Hence, in the set W , we have

$$L_1(w - \psi) \geq 0.$$

Thus, by the weak maximum principle, we obtain

$$\sup_W (w - \psi) = \sup_{\partial W} (w - \psi)^+ \leq 0 = (w - \psi)(X).$$

Thus, X is the local maximum point of $v = w - \psi$ in W so that the normal derivatives at X becomes

$$(A.17) \quad v_r - r'(\theta)v_\theta = \frac{\partial v}{\partial \nu}(X) \geq 0,$$

while the tangential derivatives at X along Γ_{shock} becomes

$$(A.18) \quad r'v_r + v_\theta = r'(w_r - \psi'd_r) + (w_\theta - \psi'd_\theta) = \frac{\partial v}{\partial t}(X) = 0.$$

Hence, we obtain

$$v_r(1 + (r')^2) \geq 0,$$

that is,

$$(A.19) \quad w_r - \psi'd_r = v_r(X) \geq 0.$$

Now, since ρ satisfies $M\rho = 0$ on Γ_{shock} with given $r'(\theta) > 0$, multiplying $e^{\mu_0 u} \bar{c}'$ throughout $M\rho = 0$, it follows from (A.18) that

$$\begin{aligned} 0 &= \sum_{i=1}^2 \beta_i e^{\mu_0 u} (\bar{c}^2)' D_i \rho \\ &= \beta_1 (w_r + 2e^{\mu_0 u} r) + \beta_2 w_\theta \\ &= 2e^{\mu_0 u} r \beta_1 + (\beta_1 - \beta_2 r') w_r + \beta_2 (\psi' d_\theta + \beta_2 \psi' r' dr). \end{aligned}$$

At X , noting that $u = 0$ and using (A.18), we obtain

$$\begin{aligned} 0 &\geq 2r\beta_1 - \psi'\mu d_r(X) + \beta_2(\psi' d_\theta + \beta_2 \psi' r' dr)(X) \\ &= 2r\beta_1 + \psi'(0)(\beta_1 - r'\beta_2) \\ &= 2r\beta_1 + \psi'(0)\mu. \end{aligned}$$

Thus, we show that, if ψ satisfies

$$\psi'(0) = \frac{m_0 b_1 + d_1 f_1}{e_0(1 - e^{-b_1 d_1/e_0})} - \frac{f_1}{b_1} < -\frac{2r\beta_1}{\mu}$$

so that the last inequality becomes negative, which is a contradiction, where we have used the fact that $\mu < 0$.

The second order Taylor series expansion of $e^{-b_1 d_1/e_0}$ is

$$e^{-b_1 d_1/e_0} = 1 - \frac{b_1}{e_0} d_1 + \frac{b_1^2}{2e_0^2} e^{-b_1 d_2/e_0} d_1^2,$$

for some $0 \leq d_2 \leq d_1$. Hence we have

$$\psi'(0) = \frac{m_0}{d_1(1-\vartheta)} + \frac{f_1}{b_3} \frac{\vartheta}{1-\vartheta},$$

where $\vartheta = O(d_1) = \frac{b_3^2}{2e_0^2} e^{-b_3 d_2/e_0} d_1$.

Now, at X , recall (A.18) and notice that

$$r'w_r + w_\theta = r'\psi'd_r + \psi'd_\theta = 0,$$

so that

$$(A.20) \quad r'(\bar{c}^2)'\rho_r - 2rr' = r'w_r = -w_\theta = -(\bar{c}^2)'\rho_\theta.$$

Also, from $M\rho = 0$, we have

$$r'(\bar{c}^2)'\rho_r = -r'(\bar{c}^2)'\frac{\beta_2}{\beta_1}\rho_\theta = (-1 + \frac{\mu}{\beta_1})(\bar{c}^2)'\rho_\theta.$$

Thus, from (A.20), we have

$$(A.21) \quad w_\theta = (\bar{c}^2)'\rho_\theta = \frac{2r'r\beta_1}{\mu}, \quad w_r = -\frac{2r\beta_1}{\mu}.$$

Notice that, for small $d_1 > 0$, we can write $m_0 = w(P)$ at some point in W and

$$\begin{aligned} m_0 = w(P) &\leq w(X) + Dw(P) \cdot (P - X) + \|w\|_{1+\alpha, W} d_1^{1+\alpha} \\ &= \frac{2r\beta_1}{\mu} d_1 + m_1 d_1^{1+\alpha}, \end{aligned}$$

where $m_1 = \|w\|_{1+\alpha, W}$. Thus, for all d_1 , we obtain

$$\psi'(0) \leq \frac{1}{1-\vartheta} \left(\frac{2r\beta_1}{\mu} + m_1 d_1^\alpha + \frac{f_1}{b_1} \vartheta \right),$$

where we recall $\vartheta = O(d_1)$. Then we have

$$2r\beta_1 + \psi'(0)\mu \leq 2r\beta_1 + \frac{\mu}{1-\vartheta} \left(\frac{2r\beta_1}{\mu} + m_1 d_1^\alpha + \frac{f_1}{b_1} \vartheta \right).$$

Therefore, we have

$$2r\beta_1 + \frac{2r\beta_1}{1-\vartheta} = \frac{2-\vartheta}{1-\vartheta} r'(\theta) (c^2(r^2 - \bar{c}^2) - 3\bar{c}^2(c^2 - r^2)) < 0 \quad \text{for } \vartheta < 1,$$

where we have used the fact $r(X) < \bar{c}(X)$. Thus, for sufficiently small d_1 and $\vartheta = O(d_1)$, we have

$$2r\beta_1 + \psi'(0)\mu < 0,$$

which is a contradiction.

Therefore, there is no such X , which implies that the set $B = \emptyset$. This completes the proof.

A.3. Proof of Lemma 3.9. For the notational simplicity, throughout the proof, we write $\rho = \rho^{\varepsilon, \delta}$. To prove the monotonicity, we argue by contradiction.

Let us first examine the C^1 -function ρ restricted to Γ_{shock} . Since $r'(\theta) > 0$, ρ is now a function of a single variable, i.e., the second component of a point $P = (r(\theta), \theta)$ on Γ_{shock} , without confusion. We can order the points along Γ_{shock} by θ and refer to the intervals along Γ_{shock} by the label. Then the lack of monotonicity implies that there exist points Θ_1 and Θ_2 on Γ_{shock} , with $P_2 < \Theta_1 < \Theta_2 < P_1$, at which $\rho(\Theta_1) > \rho(\Theta_2)$. Then we immediately deduce that

1. In (P_2, Θ_2) , there exists \tilde{C} with $\rho(\tilde{C}) = \max_{[P_2, \Theta_2]} \rho$;
2. In (\tilde{C}, P_1) , there exists D with $\rho(D) = \min_{[\tilde{C}, P_1]} \rho$.

We want to identify the points C and D on Γ_{shock} with $C < D$ such that

- (i) $\rho(P_2) \leq \rho \leq \rho(C)$ on $[P_2, C]$;
- (ii) $\rho(C) \geq \rho \geq \rho(D)$ on $[C, D]$;
- (iii) $\rho(D) \leq \rho \leq \rho(P_1)$ on $[D, P_1]$.

Now, property (ii) may not hold with $C = \tilde{C}$ because $\rho(\tilde{C})$ is the maximum value of ρ only at the interval $[P_2, D]$, and we may have $D > \Theta_2$. Then, if there is a point in (P_2, Θ_2) at which $\rho > \rho(\tilde{C})$, we let C to be the point. Otherwise, we choose $C = \tilde{C}$. Thus, all the three properties hold.

Now we look at the function ρ in Ω . The idea is to partition Ω into three subdomains by two curves Γ_C and Γ_D from C and D to the points A and B respectively on Γ_0 , in such a way $\rho(A) > \rho(B)$ that we can deduce that there is a point m on Γ_0 at which ρ obtains a maximum on either the subdomain Ω_A or the domain Ω_B , thus violating the Hopf maximum principle. This is also the case even if it happens to be the origin O . It suffices to show that $\rho(m)$ is the maximum value of ρ on the boundary of Ω_A or Ω_B .

We now construct the Lipschitz curves on which ρ has certain monotone property. That is,

$$(A.22) \quad \rho(A) \geq \rho \geq \rho(C) - \mu \quad \text{on } \Gamma_C, \quad \rho(A) > \rho(C),$$

and

$$(A.23) \quad \rho(B) \leq \rho \leq \rho(D) + \mu \quad \text{on } \Gamma_D, \quad \rho(B) < \rho(D),$$

for certain number $\mu > 0$. We specify

$$\mu = \frac{1}{4} \min\{\rho(C) - \rho(D), \rho_1 - \rho(C), \rho(D) - \bar{\rho}\}.$$

Since $\rho \in C^\alpha(\bar{\Omega})$, we have

$$|\rho(X_1) - \rho(X_2)| \leq M|X_1 - X_2|^\alpha$$

for some $M > 0$ and $X_1, X_2 \in \bar{\Omega}$. Now, on any ball with radius $r > 0$,

$$\text{Osc}(\rho) \leq 2Mr^\alpha.$$

Let $R = (\frac{\mu}{2M})^{-\alpha}$. We have

$$\text{Osc}_{B_R \cap \Omega}(\rho) \leq \mu.$$

Now we construct Γ_C as follows (as in Fig 5): In $B_R(C) \cap \Omega$, $\rho(C)$ can not be the maximum value of ρ . Thus, there exists a point in $\partial B_R(C) \cap \Omega$ where $\rho > \rho(C)$. Let X_1 be a point at which ρ attains its maximum value in $\overline{B_R(C)}$. Since the first segment of Γ_C is a straight line from C to X_1 , we have

$$\rho(X_1) > \rho(C),$$

and, on the segment,

$$\rho(X) \geq \rho(C) - \mu, \quad \rho(X) \leq \rho(X_1).$$

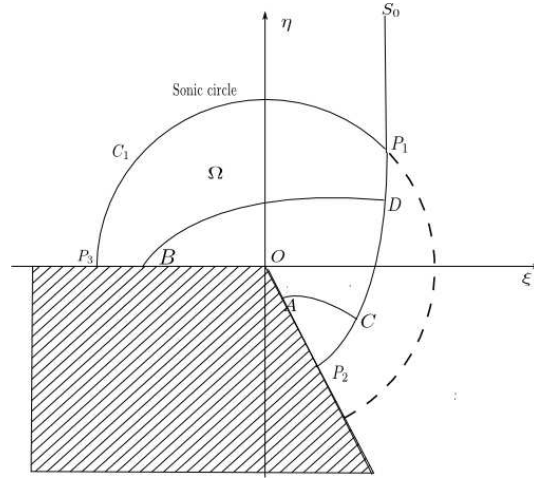


FIGURE 5. Hypothetical Curves

Now we continue inductively, forming a sequence of the line segments with corners at $\{X_i\}$ (take $X_0 = C$), along which $\rho(X) \geq \rho(C) - \mu$ and $\rho(X_1) < \rho(X_2) < \dots$. To show this fact, we let

$$\Omega_j = \Omega_\epsilon \setminus \overline{\{\cup_0^{j-1} B_R(X_i)\}}.$$

Then $X_j \in \partial\Omega_j$.

Consider $B_R(X_i)$. Note that $\rho(X_j)$ is the largest value of ρ on the part of $B_R(X_i)$ inside the complement of Ω_j . However, $\rho(X_j)$ is less than the maximum value of ρ on $B_R(X_j)$, by the mean value property. Hence, there is a point $X_{j+1} \in \partial B_R(X_i) \cap \Omega_j$ at which ρ attains its maximum value in $\overline{B_R(X_i)}$. Again, along the straight line from X_j to X_{j+1} , we have

$$\rho \geq \rho(X_j) - \mu \geq \rho(C) - \mu.$$

Now,

$$\text{dist}(X_{i-1}, \Omega_i) = R$$

and

$$\Omega_j \subset \Omega_{j-1} \subset \cdots \subset \Omega_1,$$

so that, for $k \geq i + 1$,

$$\text{dist}(X_i, \Omega_k) \geq R.$$

Since $X_k \in \partial\Omega_k$, the estimate

$$\text{dist}(X_{j+1}, X_i) \geq R \quad \text{for } i \leq j$$

follows. Hence,

$$\text{dist}(X_j, X_i) \geq R \quad \text{for } i \neq j.$$

Since the domain Ω is finite, this process must end at finite steps when we reach a point $X_L = B \in \partial\Omega$. By construction, Γ_C satisfies the properties in (A.22). Similarly, we construct Γ_D , with termination point $A \in \partial\Omega$.

We now locate A and B . We note that the two curves cannot cross each other. This is because, at every point on Γ_C ,

$$\rho \geq \rho(C) - \mu > \rho(D) + \mu;$$

while, at every point on Γ_D , we have

$$\rho < \rho(D) + \mu.$$

Furthermore, Γ_C cannot terminate at Γ_{sonic} where $\rho > \rho_1 - \mu > \rho(D) + \mu$. For the same reason, it can not come back to Γ_{shock} in $[P_2, C]$ or $[C, D]$ where $\rho \leq \rho(C)$. Finally, A cannot lie in the segment $[P_2, C]$ of Γ_{shock} , because this would trap Γ_D in a region where $\rho \geq \rho(D)$ (or more simply, this would contradict with the fact that D is not a local minimum in Ω). Hence, A has to end on Γ_0 .

Similarly, B cannot lie on Γ_{shock} where $\rho \geq \rho(D)$ in the interval $[D, P_1]$ and must lie on Γ_0 (see Fig 5).

Now we reach to our final contradiction. Since $\rho(A)$ is larger than $\bar{\rho}$ and $\rho(B)$, there is a point m along the boundary P_2OB at which ρ attains a maximum. Assume first that m is not the origin, then m can not be a local maximum for the domain Ω by the Hopf lemma. However, along the entire boundary of the domain P_2CDBP_2 , $\rho \leq \rho(m)$, which implies that it is a maximum. This is a contradiction. Now, if m coincides with O , the similar minimum point X resembling B can not coincide with O . We can find that there is no place for such X either. Thus this is also a contradiction. We conclude that ρ is monotone along Γ_{shock} from P_2 to P_1 .

A.4. Proof of Proposition 5.3. The proof is divided into two steps.

In the proof below, all the constants depend only on the data, i.e., N , c_1 , \hat{r} , R , β , and $\inf_{Q_{\hat{r},R}^+ \cap \{x > \hat{r}/2\}} \psi$, unless otherwise is stated.

Step 1: We prove that there exist $\alpha_1 \in (0, \frac{1}{2})$ and $r_1 > 0$ such that, if $W \in C(\overline{Q_{\hat{r},R}^+}) \cap C^2(Q_{\hat{r},R}^+)$ satisfies (5.20)–(5.22), then

$$(A.24) \quad W(x, y) \leq \frac{c_1(1 - \mu_1)}{r^\alpha} x^{1+\alpha} \quad \text{in } Q_{r, \frac{rR}{8}}^+,$$

whenever $\alpha \in (0, \alpha_1]$, $r \in (0, r_1]$, and $\mu_1 < \min\{\mu, \frac{1}{2}\}$.

By Lemma 5.2,

$$(A.25) \quad W(x, y) \leq c_1(1 - \mu_1)x \quad \text{in } Q_{r_0, \frac{15R}{16}}^+,$$

where r_0 depends only on the data.

Fix y_0 with $|y_0| \leq \frac{7R}{8}$. We now prove that

$$W(x, y_0) \leq \frac{c_1(1 - \mu_1)}{r^\alpha} x^{1+\alpha} \quad \text{for } x \in (0, r).$$

As in [2], without loss of generality, we may assume that $R = 2$ and $y_0 = 0$. Otherwise, we set $\tilde{W}(x, y) = W(x, y_0 + \frac{R}{32}y)$ for all $(x, y) \in Q_{\hat{r}, 2}^+$. Then $\tilde{W}(x, y) \in C(\overline{Q_{\hat{r}, R}^+}) \cap C^2(Q_{\hat{r}, R}^+)$ satisfies (5.20)–(5.22), with some modified constants N , ϑ , and O_i , depending only on the corresponding quantities in the original equation and on R . We conclude that, without loss of generality, we can assume that $y_0 = 0$ and $R = 2$. That is, it suffices to prove that

$$(A.26) \quad W(x, 0) \leq \frac{c_1(1 - \mu_1)}{r^\alpha} x^{1+\alpha} \quad \text{for } x \in (0, r),$$

for some $r \in (0, r_0)$ and $\alpha \in (0, \alpha_1)$, under the assumptions that (5.20)–(5.22) hold in $Q_{\hat{r}, 2}^+$ and (A.25) holds in $Q_{r_0, 2}^+$.

For any given $r \in (0, r_0)$, let

$$(A.27) \quad A_1 r = c_1(1 - \mu_1), \quad B_1 = c_1(1 - \mu_1),$$

$$(A.28) \quad v = A_1 x^{1+\alpha}(1 - y^2) + B_1 x y^2.$$

Since (5.21) holds on $Q_{\hat{r}, 2}^+ \cap \{x = 0\}$ and (A.25) holds in $Q_{r_0, 2}^+$, then, for all $x \in (0, r)$ and $|y| \leq 1$, we obtain

$$\begin{cases} v(0, y) = 0 = W(0, y), \\ v(r, y) = (A_1 r^\alpha(1 - y^2) + B_1 y^2)r = c_1(1 - \mu_1) \geq W(r, y), \\ v(x, \pm 1) = c_1(1 - \mu_1)x \geq W(x, \pm 1). \end{cases}$$

Thus, we have

$$(A.29) \quad W \leq v \quad \text{on } \partial Q_{r_0, 1}^+.$$

From (5.20), we obtain

$$\mathcal{L}_2 v - \mathcal{L}_2 W = \mathcal{L}_2 v - (c_1 O_2 + c_1^2 O_3).$$

In order to rewrite the right-hand side in a convenient form, we write the term v_{yy} in the expression of $\mathcal{L}_2 v$ as $(1 - y^2)v_{yy} + y^2 v_{yy}$. Then a direct calculation yields

$$\mathcal{L}_2 v - (c_1 O_2 + c_1^2 O_3) = c_1 x^\alpha (1 - y^2) J_1 + y^2 J_2,$$

where

$$\begin{aligned}
J_1 = & \alpha(\alpha+1)c_1A_1 + \alpha(\alpha+1)A_1^2x^\alpha(1-y^2) + \alpha(\alpha+1)A_1\frac{O_1}{x} - (\alpha+1)c_1A_1 \\
& + (\alpha+1)(O_2 + 2c_1O_3) + (\alpha+1)^2A_1^2x^\alpha - (\alpha+1)^2A_1^2x^\alpha O_3x^\alpha \\
& + 2(\alpha+1)A_1B_1x^\alpha O_3y^2 - 2A_1x + 2B_1x^{1-\alpha} + O_4(-2A_1x + 2B_1x^{1-\alpha}) \\
& - c_2\frac{O_2}{x^\alpha} - 2c_1^2\frac{O_3}{x^\alpha},
\end{aligned}$$

and

$$\begin{aligned}
J_2 = & \alpha(\alpha+1)A_1B_1x^\alpha - 2c_1B_1 + B_1^2y^2 + 2(\alpha+1)A_1B_1x^\alpha(1-y^2) - B_1^2O_3y^2 \\
& + O_4(-2A_1x^{\alpha+1} + 2B_1x) + 2B_1x^{\alpha+1} + 4\left(\frac{1}{(\gamma-1)c_1^2} - O_5\right)(A_1x^{\alpha+1} + B_1x)^2.
\end{aligned}$$

By (5.8) and (A.27), we obtain

$$(A.30) \quad J_1 \leq (\alpha-1)(\alpha+1)c_1A_1 + 2C(2-\mu)r^{1-\alpha},$$

$$(A.31) \quad J_2 \leq -B_1c_1 + B_1^2 + Cr^\alpha.$$

Moreover,

$$\begin{aligned}
(A.32) \quad & \mathcal{L}_2(v-W) - v_{xx}(v-W) \\
& = (c_1x + W + O_1)(v-W)_{xx} - (c_1 - O_2 - 2c_1O_3)(v-W)_x \\
& \quad + (1 - O_3)(v+W)_x(v-W_x + (1+O_4)(v-W)_{yy}) \\
& \quad - \left(\frac{1}{(\gamma-1)c_1^2} - O_5\right)(v+W)_x(v-W)_x \\
& \leq c_1x^\alpha((\alpha-1)(\alpha+1)c_1A_1 + 2C(2-\mu)r^{1-\alpha})(1-y^2) + (-B_1c_1 + B_1^2 + Cr^\alpha)y^2 \\
& \quad - (1+\alpha)\alpha A_1(A_1x^{1+\alpha}(1-y^2) + B_1xy^2)(1-y^2) \\
& \quad + (1+\alpha)\alpha c_1(1-\mu_1)A_1x^\alpha(1-y^2) \\
& \leq c_1x^\alpha((2\alpha-1)(\alpha+1)c_1A_1 + 2C(2-\mu)r^{1-\alpha})(1-y^2) \\
& \quad + (-B_1c_1 + B_1^2 + Cr^\alpha)y^2 \\
& = c_1x^\alpha(1-y^2)I_1 + y^2I_2,
\end{aligned}$$

where we have used the fact that

$$v_{xx}(W-v) = \alpha(\alpha+1)A_1x^{\alpha-1}(1-y^2)(W-v) \leq (1+\alpha)\alpha c_1(1-\mu_1)A_1x^\alpha(1-y^2).$$

Then we can choose $\alpha_1 > 0$, depending only on μ_1 , so that, if $0 < \alpha \leq \alpha_1$,

$$(A.33) \quad (2\alpha-1)(\alpha+1)c_1A_1 < -\frac{\mu_1}{2}.$$

Such a choice of $\alpha_1 > 0$ is possible because we have the strict inequality in (A.33) when $\alpha = 0$, and the left-hand side is an increasing function of $\alpha > 0$. Now, choosing

$r_1 > 0$ so that

$$(A.34) \quad r_1 < \min\left\{\left(\frac{\mu_1}{4c_1}\right)^{\frac{1}{\alpha}}, \left(\frac{B_1c_2 - B_1^2}{C}\right)^{\frac{1}{\alpha}}, r_0\right\}$$

is satisfied, we use (A.30)–(A.34) to obtain

$$(A.35) \quad I_1 < 0, \quad I_2 < 0 \quad \text{in } Q_{r,1}^+.$$

Then, by (A.32), we obtain

$$(A.36) \quad \mathcal{L}_2(v - W) - v_{xx}(v - W) < 0 \quad \text{in } Q_{r,1}^+$$

whenever $r \in (0, r_1]$ and $\alpha \in (0, \alpha_1]$. Then $v - W$ satisfies the maximum principle in $Q_{r,1}^+$, which implies

$$(A.37) \quad W \leq v, \quad \text{in } Q_{r,1}^+.$$

In particular, using (A.27) and (A.28) with $y = 0$, we arrive at (A.24).

Step 2. We now establish Proposition 5.3. As argued before, without loss of generality, we may assume that $R = 2$ and it suffices to show that

$$(A.38) \quad W(x, 0) \leq Ax^{1+\alpha} \quad \text{for } x \in [0, r].$$

By Step 1, it suffices to prove that (5.23) holds for the case $\alpha > \alpha_1$. Fix any $\alpha \in (\alpha_1, 1)$ and set the following comparison function:

$$(A.39) \quad v = \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}} x^{1+\alpha} (1 - y^2) + \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1}} x^{1+\alpha_1} y^2.$$

By Step 1,

$$(A.40) \quad W \leq v \quad \text{on } \partial Q_{r,1}^+ \text{ for } r \in (0, r_1].$$

As in the proof of Step 1, we write

$$\mathcal{L}_2(v - W) = (1 + \alpha) \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}} x^\alpha (1 - y^2) \hat{J}_1 + (1 + \alpha_1) \frac{c_1(1 - \mu_1)}{r_1^\alpha} x^{\alpha_1} y^2 \hat{J}_2,$$

where

$$\begin{aligned} \hat{J}_1 &= \alpha c_1 + \alpha \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}} x^{1+\alpha} (1 - y^2) + \alpha \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1}} x^{\alpha_1} y^2 + \alpha \frac{O_1}{x} - c_1 \\ &\quad + O_2 + 2c_1 O_3 + (1 + \alpha)(1 + O_3) \frac{c_1(1 - \mu_1)}{r_1^{\alpha_1} r^{\alpha - \alpha_1}} x^\alpha (1 - y^2) \\ &\quad + 2(1 + O_3)(1 + \alpha_1) \frac{c_1(1 - \mu_1)}{r_1^\alpha} x^{\alpha_1} y^2 - \frac{2(1 + O_4)}{\alpha + 1} x + \frac{2(1 + O_4)r^{\alpha - \alpha_1}}{\alpha + 1} x \\ &\quad + \frac{1}{\alpha + 1} \frac{r_1^{\alpha_1} r^{\alpha - \alpha_1}}{1 - \mu_1} x^{1-\alpha} \left(\frac{O_2}{x} + c_1 \frac{O_3}{x} \right), \end{aligned}$$

and

$$\begin{aligned}
\hat{J}_2 = & \alpha_1 c_2 + \alpha_1 \frac{c_2(1-\mu_1)}{r_1^{\alpha_1} r^{\alpha-\alpha_1}} x^\alpha (1-y^2) \alpha_1 \frac{c_2(1-\mu_1)}{r_1^{\alpha_1}} x^{\alpha_1} y^2 + \alpha_1 \frac{O_1}{x} - c_2 + O_2 + 2c_2 O_3 \\
& + (1+\alpha_1) \frac{c_2(1-\mu_1)}{r_1^{\alpha_1}} x^{\alpha_1} y^2 - (1+\alpha_1) O_3 \frac{c_2(1-\mu_1)}{r_1^{\alpha_1}} x^{\alpha_1} - \frac{2(1+O_2)}{r^{\alpha-\alpha_1}} x^{1+\alpha-\alpha_1} \\
& + \frac{2(1+O_2)}{\alpha_1+1} x + 4 \left(\frac{1}{(\gamma-1)c_2^2} - O_5 \right) \frac{c_2(1-\mu)}{r_1^{\alpha_1}} x^{2+\alpha_1} \left(-\frac{1}{r^{\alpha-\alpha_1}} x^{\alpha-\alpha_1} + 1 \right)^2 \\
& - \frac{1}{1+\alpha_1} \frac{r_1^\alpha x^{1-\alpha_1}}{1-\mu_1} \left(\frac{O_2}{x} + c_2 \frac{O_3}{x} \right).
\end{aligned}$$

It is easy to see that $\hat{J}_1, \hat{J}_2 < 0$. By continuity, we can choose $r > 0$ sufficiently small, depending only on $N, c_1, \hat{r}, R, \vartheta$, and α_1 . Thus, we obtain

$$\begin{aligned}
\mathcal{L}_2(v-W) - v_{xx}(v-W) \leq & (1+\alpha) \frac{c_1(1-\mu_1)}{r_1^{\alpha_1} r^{\alpha-\alpha_1}} x^\alpha (1-y^2) ((\alpha-1)c_1 + Cr^{\alpha_1}) \\
& + (1+\alpha_1) \frac{c_1(1-\mu_1)}{r_1^{\alpha_1}} x^{\alpha_1} y^2 ((\alpha_1-1)c_1 + Cr^{\alpha_1}) \\
& + \alpha(1+\alpha) \frac{c_1^2(1-\mu_1)^2}{r_1^{2\alpha_1} r^{\alpha+\alpha_1}} x^{\alpha+\alpha_1} (1-y^2) \\
& + \alpha_1(1+\alpha_1) \frac{c_1^2(1-\mu_1)^2}{r_1^{2\alpha_1}} x^{2\alpha_1} (1-y^2) \quad \text{in } Q_{r,1}^+,
\end{aligned}$$

where we have used the result in Lemma 5.3 that $W \leq Bx^{1+\alpha_1}$. It is easy to check that

$$\mathcal{L}_2 v - \mathcal{L}_2 W - v_{xx}(v-W) < 0.$$

As the proof in Lemma 5.3, it is easy to prove that (5.23) holds with

$$A = \frac{c_1(1-\mu_1)}{r_1^{\alpha_1} r^{\alpha-\alpha_1}}.$$

This complete the proof.

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